Efficiency in an overlapping generations setting with endogenous fertility∗

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FEBRUARY 2004

March 31, 2004

Abstract

In this paper we extend the notion of Pareto efficiency to overlapping generations settings in which fertility decisions are endogenous. The contribution of this paper is threefold. First, we provide two definitions of Pareto dominance and, consequently, of Pareto efficiency referred to Millian efficiency and strong efficiency, differing in the treatment of agents that never get to be born. Second, we provide necessary (static) and sufficient (dynamic) conditions to determine whether or not an allocation is Millian efficient. Third, we characterize Millian efficient allocations as the equilibria of a decentralized price mechanism, and enunciate the Fundamental Welfare Theorems in an environment with endogenous fertility.

Keywords: Endogenous fertility, Pareto optimality, dynamic efficiency.

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1 Introduction

This paper studies the issue of Pareto efficiency in an overlapping generations setting with endogenous population. In an environment in which the set of agents is endogenous, we adopt a weak extension of the notion of Pareto efficiency, referred to as Millian efficiency. The notion of Pareto dominance underlying the notion of Millian efficiency is based exclusively on preferences of those agents alive, and allows only for welfare comparisons of symmetric allocations (i.e., allocations in which all living individuals of the same generations take the same decisions). We provide necessary conditions that every Millian efficient allocation must satisfy, and a sufficient condition determining whether a given allocation satisfying these necessary conditions is Millian efficient. With these results at hand, we characterize Millian efficient allocations as the equilibria of a decentralized price mechanism. Finally, we discuss an alternative extension of the Pareto criterium that strengthens the Millian notion, referred to as strong efficiency.

Several papers in the literature provides necessary and sufficient conditions for Pareto efficiency in dynamic economies. The first complete characterization was provided by Cass (1972) in the context of a simple physical capital growth model. Later, Balasko and Shell (1980) focused on an overlapping generations exchange economy to show that, despite an allocation may be short-run efficient (or statically efficient), i.e., it cannot be improved upon by a reallocation of resources of a finite number of generations, it might not be long-run efficient (or dynamically efficient), that is, fully efficient. Other relevant extensions of Cass’ results are those obtained by Galor and Ryder (1991), Chattopadhyway and Gottardi (1999) or Molina and Pintos (2003). All these extensions, however, deal with exogenous fertility.

In the literature of overlapping generation economies with endogenous fertility, two different approaches to provide normative principles can be distinguished: a first approach identifies socially optimal allocations with those solving a certain class of social welfare maximization problems; while a second approach focuses only on the steady state optimal allocations (also referred to as golden rule allocations), that is, allocations that maximize the utility obtained by a representative consumer in each generation among those feasible, stationary allocations. Neither one of these two approaches takes explicitly into account the problem of dynamic efficiency, nor the fact that the standard Pareto criterium is not straightforward applicable to environments in which the set of agents is endogenous. In such environments, extensions of the notion of Pareto efficiency requires a careful assessment of how people who might never get to be born should be considered for welfare comparisons between any pair of allocations. An exception within the literature of endogenous fertility is the recent work by Golosov et al (2004), who analyze two extensions of the notion of Pareto efficiency (referred to as $P-$efficiency and $A-$ efficiency) in a general equilibrium framework with endogenous population, and show that allocations solving a certain type of social welfare maximization problems form a particular subclass of the class of $P-$efficient allocations. However, the notions of Pareto dominance underlying these two notions of efficiency (which differ from the notion of Millian efficiency proposed in this paper) may be deemed undesirable from symmetry considerations, as we shall argue throughout the paper. Furthermore, they assume that the potential set of agents is discrete, and this assumption brings with it considerable difficulties if one is concerned with identifying efficient allocations in overlapping generations settings with non-altruistic agents.

In this paper, we show that exploring the properties of Millian efficient allocations in over-

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1 See e.g., Nerlove, Razin and Sadka (1986), Cigno (2002), Groezen et al (2003) or Razin and Sadka (1995, Ch.5) for a survey.

lapping generations framework is fairly simple, at least with respect to necessary conditions. In every Millian efficient allocation, the rate of return to all investments (including investment on children) should be equal to the marginal rate of substitution between current consumption and future consumption. Identifying sufficient conditions guaranteeing that a given allocation is Millian efficient is a more difficult task. When fertility is endogenous, the set of feasible allocations faced by agents in overlapping generations economies is non-convex, and the sufficient conditions for dynamic efficiency provided by Cass (1972) and Balasko and Shell (1975) cannot be applied. Thus, we provide an extension of Balasko and Shell’s sufficient condition to non-convex settings.

With this results at hand, we re-write some well known results of welfare economics in terms of Millian efficiency. First, we show that the set of allocations that maximize a particular class of social welfare functions, referred to as Millian social welfare functions, form a proper subset of the set of Millian efficient allocations. Second, we adapt the Fundamental Theorems of Welfare Economics to a setting with endogenous population by characterizing every statically Millian efficient allocation as the equilibrium of a decentralized sequential price mechanism. Similar to the case of economies with exogenous population, every Millian efficient can be decentralized by initially selecting an appropriate sequence of intergenerational transfers, and then allowing the agents to determine their consumption and investment decisions at competitive markets. Differently from the standard, exogenous population case, an incentive scheme that links intergenerational transfers with fertility decisions is needed. More precisely, for every system of intergenerational transfers that achieves Millian efficiency, every young adult has to pay a lump-sum tax (or, in some cases, receive a lump-sum subsidy), and every old adult will receive a subsidy (or pay a tax) which depends linearly of the number of children she decided to have. As a particular case, we also show that the allocation corresponding to a decentralized equilibrium with no intergenerational transfers (for which there is no need to subsidize or tax children) is (statically) Millian efficient. In contrast with other environments with incomplete markets, this particular case shows that the absence of a market (in this case, a market where offspring may bargain with their parents the right to be born) does not yield any efficiency loss, at least if one is concerned with Millian efficiency.

To conclude the paper, we discuss the difficulties of further extensions of the Pareto criterium to settings with endogenous populations. Most of this discussion focuses on a criterium of efficiency that strengthens the Millian notion, which we refer to as strong efficiency. This criterium can be regarded as an adaptation to overlapping generation settings, like the one studied throughout the paper, of the notion of $P-$efficiency proposed by Golosov et al (2004), and it differs from the Millian notion in two respects. First, agents that might never get to be born are taken into account in social welfare judgements. This is done by specifying a utility threshold above which it is assumed that any agent is better off by being born (and obtains utility above that threshold) than otherwise. Second, the criterium allows one to compare symmetric allocations with asymmetric ones. Thus, a symmetric Millian efficient allocation is strongly efficient if is not Pareto dominated by any other allocation including asymmetric ones, where the notion of Pareto dominance refers to all potential agents in the economy.

We show that the notion of strong efficiency involves an important difficulty: if we focus on stationary allocations in an economy without growth, then the only Millian efficient, symmetric allocations are those at which the agents obtain their utility threshold. We use this fact to argue that strong efficiency might be deemed undesirable from symmetry considerations, and suggest other possible extensions.

The paper is organized as follows. In section 2, we introduce the model. Next (Sections 3 and 4) we present the notion of Millian efficiency mentioned above and provide necessary and sufficient
conditions to determine whether an allocation is efficient in this sense. In section 5 we characterize Millian efficient allocations as the equilibria of a decentralized sequential price mechanism, and use this characterization to study whether or not missing markets originate market failures under the Millian notion of efficiency. In Section 6, we provide an alternative notion of efficiency, referred to as strong efficiency, and explores the properties of allocations satisfying this criterion. Finally, section 7 presents the main conclusions of the paper and discusses possible extensions.

2 The Model

Consider an overlapping generations economy in which each generation is formed by a set of identical agents whose size is determined endogenously. Agents live for three periods of time. At each date \( t = 0, 1, 2 \ldots \) and, for each dynasty, there exist \( n_{t-2} \in \mathbb{R}_+ \) old adults (that is, \( n_{t-2} \) agents born at date \( t - 2 \)), \( n_{t-1} \) middle-age adults (that is, \( n_{t-1} \) agents born at date \( t - 1 \)) and \( n_t \) children (that is, \( n_t \) agents born at \( t \)). The number of old adults at \( t = 0 \) is normalized to one, and the number of young adults at \( t = 0 \) is given by the initial condition \( N_{-1} = n_{-1} = \pi_{-1} > 0 \). Hence, the number of agents born at any period \( t \), denoted by \( N_t \), is given by

\[
N_t = \prod_{\tau = -1}^{t} n_{\tau} = \pi_{-1} \prod_{\tau = 0}^{t} n_{\tau},
\]

At each period \( t = 0, 1, 2 \ldots \), a perishable good is produced through a technology using human capital \((H_t)\) and physical capital \((K_t)\) accumulated from previous period \( t - 1 \) as inputs,

\[
Y_t = F(K_t, H_t)
\]

where \( F_t(\cdot) \) is a concave, constant return to scale production function. By letting \( y_t = \frac{Y_t}{N_{t-1}} \), \( h_t = \frac{H_t}{N_{t-1}} \) and \( k_t = \frac{K_t}{N_{t-1}} \), \( Y_t \) can be equivalently written, in per worker terms, as

\[
y_t = F(k_t, h_t)
\]

where \( k_t \) represents physical capital and \( h_t \) human capital. The middle-age adults are endowed with one unit of time to work, which is supplied inelastically at each period \( t = 0, 1, 2 \ldots \). Young agents (children) are born with an endowment of \( h^y_t \) units of basic knowledge (an exogenous variable), which is used to produce their human capital according to

\[
h_{t+1} = H(d_t, h^y_t)
\]

where \( d_t \) denotes the physical resources invested in the education of each agent born at \( t \) and \( H(\cdot) \) is a non-decreasing, concave production function.

To complete the description of technologies, it is assumed that physical capital is fully depreciated in the production process, and that \((k_0, h_0)\) is given by the initial condition

\[
(k_0, h_0) = (k_0, h_0).
\]

Finally, it is also assumed that rearing children is costly in terms of the homogeneous good. More precisely, for each \( t = 1, 2, \ldots \) per capita costs \( b(n_t) \) of rearing \( n_t \) children are determined by a non-decreasing, convex function \( b : \mathbb{R}_+ \to \mathbb{R} \).

The aggregate output of the homogeneous good is used to finance aggregate investments in both human and physical capital, denoted respectively by \( D_t \) and \( K_{t+1} \), to cover costs of rearing
children (given by \( N_{t-1} b(n_t) \)), and to finance aggregate consumption by old adults (denoted by \( C^o_t \)) and by young adults (denoted by \( C^m_t \)). In any period, the aggregate resource constraint is

\[
C^o_t + C^m_t + N_{t-1} b(n_t) + D_t + K_{t+1} \leq F(H_t, K_t),
\]

which, by letting \( c^o_t = \frac{C^o_t}{N_{t-1}} \), \( c^m_t = \frac{C^m_t}{N_{t-1}} \), and \( d_t = \frac{D_t}{N_t} \), can be equivalently written, in per worker terms, as

\[
c^o_t + n_{t-1} [c^m_t + b(n_t) + n_t (d_t + k_{t+1})] \leq F(h_t, k_t)
\]

where \((n_{-2}, n_{-1}, k_0, h_0)\) is given by the initial condition

\[
(n_{-2}, n_{-1}, k_0, h_0) = (1, \pi_{-1}, \bar{K}, \bar{h}_0)
\]

Let \( a = \{(c^m_t, c^o_t, n_t, d_t, k_{t+1}, h_{t+1})\}_{t=0}^\infty \) be an allocation, where we denote \( c^m \) the consumption in their middle-age agent’s consumption, \( c^o \) the agent’s consumption when old, and \( n \) the number of descendants that they decide to have.

For each agent born at \( t = -1 \), preferences on the consumption set \( A \) are represented by a monotone continuous utility function \( U_{-2} : A \to \mathbb{R} \) defined, for each \( a \in A \), by \( U_{-2}(a) = c^o_0 \), where \( c^o_0 \) denotes the agent’s consumption at period \( t = 0 \). For each agent born in period \( t-1 \) with \( t = 0, 1, 2, ... \), preferences are represented by a continuous utility function \( U_{t-1} : A \to \mathbb{R} \) defined, for each \( a \in A \), by \( U_{t-1}(a) = u(c^m_t, c^o_{t+1}, n_t) \). We assume that the function \( u : \mathbb{R}_+^3 \to \mathbb{R} \) is non-decreasing and yields indifference curves that are strictly convex to the origin.

Throughout most of the paper, we restrict attention on feasible symmetric allocations, that is, on sequences \( a = \{(c^m_t, c^o_t, n_t, d_t, k_{t+1}, h_{t+1})\}_{t=0}^\infty \) satisfying the resource constraint in (2) and the initial condition in (3). Denote by \( \mathcal{A} \) the set containing all feasible, symmetric allocations.

3 Millian efficiency

In order to extend the notion of Pareto-dominance to a framework in which population is endogenous, we must decide first how agents that never get to be born in some allocations but do in others should be taken into account when making social welfare judgements. In this section, we propose a notion of Pareto dominance (and, hence, of Pareto efficiency) which: i) compares only symmetric allocations, and ii) it is based exclusively on preferences of the representative agent of each generation.

**Definition 1** A feasible allocation \( a \) is said to be Millian efficient (or simply, \( \mathcal{M} \)-efficient) if there does not exist another feasible allocation \( a' \) such that:

i) for all \( t = 0, 1, 2, ... \) one has

\[
U_{t-1}(a') \geq U_{t-1}(a); \text{ and}
\]

ii) there exists at least one period \( \tau \) such that

\[
U_{\tau-1}(a') > U_{\tau-1}(a)
\]

is satisfied.

The term “Millian efficiency” refers to the fact that it is based on a notion of Pareto dominance that does not use information on the number of individuals living in every period and, therefore, can be seen as an ordinalist version of a form of utilitarianism, called average utilitarianism, often associated to J.S. Mill.\(^3\) This form of utilitarianism postulates that welfare judgments involving

\(^3\)See Razin and Sadka (1995, ch 5).
different generations should be independent of the population size of each generation. The term 
Millian refers also to the fact that this efficiency criterium provides a justification of the so called 
“Millian social welfare functions,” as we shall argue later on. Nevertheless, it is important to 
point out that our approach is more general than those that identify an optimal allocation with 
either the golden rule allocation or with some particular solution to a social welfare maximization 
problem.

Finally, observe also that the notion of Pareto dominance underlying the notion of Millian 
efficiency allows one to compare only symmetric allocations. This makes our notion different from 
the notion of \( A-e \)-efficiency proposed by Golosov et al (2003). Like the Millian notion, the notion 
of Pareto dominance underlying this criterium compares any two allocations using information of 
pREFERENCES those agents who are alive under the two allocations. Unlike the Millian notion, no 
restriction is placed on the set of allocations that are feasible and, hence, comparable.

3.1 Necessary conditions. Static \( \mathcal{M} \)-efficiency

The following analysis will be restricted to symmetric allocations that yield strictly positive 
consumption to all generations living in every period \( t \); that is, to the set \( \mathcal{A} \) formed by all 
allocations \( a \in \mathcal{A} \) such that \( x_t = (c_t^m, c_{t+1}^n, n_t) > 0 \) and \( y_t = F_t(k_t, h_t) > 0 \) for all \( t \geq 0 \).

Some additional notation is introduced first. For any \( a \) let \( c_t \) be defined as the amount of per-
worker physical resources produced at \( t \) by an agent born in period \( t-1 \), which are not devoted to 
feed the old generation born at \( t-2 \), that is, \( c_t = c_t^n + b(n_t) + n_t(d_t + k_{t+1}) \).

Also, for each \( t = 0, 1, 2, \ldots \), let \( \phi : \mathbb{R}_+^2 \to \mathbb{R} \) be defined, for every pair \((y_{t+1}, h_t^n) \in \mathbb{R}_+ \), by

\[
\phi(y_{t+1}, h_t^n) = \min_{(k_{t+1}, d_t)} \{ k_{t+1} + d_t : F_{t+1}(k_{t+1}, H(d_t, h_t^n)) \geq y_{t+1} \}.
\]

That is, each cost function \( \phi \) determines, for each \( y_{t+1} \), the amount of the physical good \( \phi(y_{t+1}, h_t^n) \) 
that is required to produce \( y_{t+1} \) units of output. With this notation, the feasibility condition in 
can be equivalently written as

\[
c_t^n + n_{t-1} \left[ c_t^n + b(n_t) + n_t \phi(y_{t+1}, h_{t+1}^{n-1}) \right] \leq y_t.
\]

It should be noticed that since we have assumed that both production functions are non decreasing, 
constant returns to scale and concave functions, the cost function \( \phi(\cdot) \) must be a non decreasing, 
convex function on \( \mathbb{R}_+^2 \). Furthermore, if \( H \) exhibits constant returns to scale, then \( \phi(\cdot) \) must be 
homogeneous of degree 1. Finally, if both \( F \) and \( H \) are differentiable functions and the triple 
\((k_{t+1}, d_t, y_{t+1})\) satisfies \( F(k_{t+1}, H(d_t, h_t^n)) = y_{t+1} \) one must have

\[
\frac{1}{\phi_1(y_{t+1}, h_t^n)} = \frac{1}{F_1(k_{t+1}, H(d_t, h_t^n))} = \frac{1}{F_2(k_{t+1}, H(d_t, h_t^n))H_1(d_t, h_t^n)}.
\]

The following proposition provides necessary conditions for Millian efficiency.

Proposition 2 Every \( \mathcal{M} \)-efficient allocation \( \hat{a} \in \mathcal{A} \) verifies, for every \( t \geq 0 \),

\[
\max_{(x_t, d_t, k_{t+1}) \in \mathbb{R}_+^2} \left\{ u(x_t) : c_t^n + b(n_t) + n_t(d_t + k_{t+1}) \leq \hat{c}_t; \quad F(k_{t+1}, H(d_t, h_t^n)) = y_{t+1} \right\}.
\]
or, equivalently,

$$u(\hat{x}_t) = \max \{ u(x_t) : c_i^n + b(n_t) + n_t\phi(y_{t+1}, h_t^y) \leq \hat{e}_t;$$ \hspace{1cm} (5) $$

$$y_{t+1} - \frac{\hat{c}_{t+1}}{n_t} \geq \hat{e}_{t+1} \}.$$ 

Proof. By contradiction. Suppose that \( \hat{a} \) is an efficient allocation, and suppose there exists a period \( \tau \) for which the 5-upla \((\hat{x}_\tau, \hat{d}_\tau, \hat{k}_{\tau+1})\) corresponding to the allocation \( \hat{a} \) is not a solution to the optimization problem (4). Let now \((\overline{x}_\tau, \overline{d}_\tau, \overline{k}_{\tau+1})\) be a particular solution to such problem for \( t = \tau \). Because by assumption \( \hat{a} \) is an efficient allocation we have that \( u(\overline{x}_\tau) \geq u(\hat{x}_\tau) \), which implies that either \( b(\hat{n}_\tau) + \hat{n}_\tau(\hat{d}_\tau + \hat{k}_{\tau+1}) > \hat{e}_\tau \) or \( F(\hat{k}_{\tau+1}, \hat{h}_{\tau+1}) - \hat{c}_{\tau+1}/\hat{n}_\tau < \hat{c}_{\tau+1} \) is satisfied, which is impossible by feasibility of the allocation \( \hat{a} \). Therefore, it must be the case that \( u(\hat{x}_\tau) < u(\overline{x}_\tau) \) is satisfied. But then let \( \overline{\tau} \) be the allocation obtained from \( \hat{a} \) by replacing the term \((\hat{x}_\tau, \hat{d}_\tau, \hat{k}_{\tau+1})\) by the solution \((\overline{x}_\tau, \overline{d}_\tau, \overline{k}_{\tau+1})\) to the optimization problem in (4). Clearly, such allocation \( \overline{\tau} \) Pareto dominates \( \hat{a} \), a contradiction that establishes 2. \( \square \)

Note that \( u(\hat{x}_t) \) gives us the maximum utility that an individual born at \( t - 1 \), endowed with \( \hat{e}_t \) units of physical resources, can obtain without diminishing the resources available for the next generation. In view of this we will adopt the terminology proposed by Balasko and Shell (1980) and we will refer to an allocations satisfying the necessary conditions in Proposition 2 as a statically \( M \)-efficient allocation.

The following corollary uses the first order conditions associated to the sequence of optimization problems (4) in the statement of Proposition 2 to provide an equivalent representation of the necessary conditions that a \( M \)-efficient allocation must satisfy.

Corollary 3. For every statically efficient allocation \( \hat{a} \in A^I \), there exists a sequence \( \{R_{t+1}\}_{t=0}^\infty \) of strictly positive real numbers satisfying, for every \( t \geq 0 \),

$$R_{t+1} = \frac{u_1'(\hat{x}_t)}{u_2'(\hat{x}_t)} = \frac{1}{\phi_1(\hat{y}_{t+1}, h_t^y)} = \frac{\hat{c}_{t+1}}{n_t} = \frac{\hat{y}_{t+1} - \frac{\hat{c}_{t+1}}{n_t}}{b'(\hat{n}_t) - \frac{u_1'(\hat{x}_t)}{u_1'(\hat{x}_t)} + \left(\hat{k}_{t+1} + \hat{d}_t\right)}.$$ \hspace{1cm} (6) 

Observe that the term at the right hand side of (6) can be regarded as the rate of return to investments in children. Thus, Corollary 3 establishes that in an efficient allocation, the marginal rate of return to investments in children must be equal to the marginal rates of return corresponding to any other investment, that in turn must be equal to the marginal rate of substitution between current and future consumption.

Remark. A Millian social welfare function can be defined as a function \( \mathcal{V} : A^I \rightarrow \mathbb{R} \) defined, for every \( a \in A^I \), by \( \mathcal{V}(a) = \sum_{t=-\infty}^{\infty} \alpha^{t-1} U_{t-1}(a) \), where the constant \( \alpha \in (0, 1) \) is regarded as an intergenerational discount factor. It is straightforward to show that any allocation maximizing a Millian welfare function must satisfy the necessary conditions in (6), although the converse is not necessarily true.

3.2 Sufficient conditions. Dynamic \( M \)-efficiency.

In the previous subsection we have obtained the necessary conditions for Millian efficiency, but those conditions do not guarantee that an allocation solving the sequence of optimization problems described above is actually \( M \)-efficient. As in other overlapping generation models, an
allocation \( a \in A \) might solve the sequence of optimization problems in (4) and fail to be efficient. In the literature, this issue has been referred to as the problem of dynamic efficiency.

Well known examples of articles studying the issue of dynamic efficiency in overlapping generation models are those by Cass (1972) in the context of a simple physical capital growth model, and Balasko and Shell (1980) who focus on an overlapping generations exchange economy. All these papers show that, despite an allocation can be short-run efficient (or statically efficient), i.e., it cannot be improved upon by a reallocation of resources of a finite number of generations, it might not be long-run efficient (or dynamically efficient), that is, fully efficient. In this subsection, we extend these previous results to an environment of endogenous population.

### 3.2.1 The indirect utility function.
Initially, we introduce some notation. Let \( W_{t-1} : \mathbb{R}^2_+ \to \mathbb{R} \), the indirect utility function of generation \( t - 1 \), be defined, for every \( t \geq 0 \) and every pair \((\hat{e}_t, \hat{e}_{t+1})\), by

\[
W_{t-1}(\hat{e}_t, \hat{e}_{t+1}) = u(x_t(\hat{e}_t, \hat{e}_{t+1}))
\]

where \( x_t(\hat{e}_t, \hat{e}_{t+1}) = (e_t^m(\hat{e}_t, \hat{e}_{t+1}), e_{t+1}(\hat{e}_t, \hat{e}_{t+1}), n_t(\hat{e}_t, \hat{e}_{t+1})) \) is a solution to the optimization problem (4) in the statement of Proposition 2. That is, for each \((\hat{e}_t, \hat{e}_{t+1}) \in \mathbb{R}^2\), the function \( W_{t-1} \) at each \( t \) determines the maximum utility achievable by an agent born at time \( t - 1 \) with the (per worker) resources of the homogeneous good given by \( \hat{e}_t \), when the agent is constrained to provide the following generation with at least \( \hat{e}_{t+1} \) units of the homogeneous good.

Note that the two inequality constraints in the optimization problems in (4) or (5) must be binding for any solution \( x_t(\hat{e}_t, \hat{e}_{t+1}) \) to (4). Thus, these constraints can be represented equivalently as

\[
e^m_t + b(n_t) + n_t \phi \left( \frac{c^o_{t+1}}{n_t} + \hat{e}_{t+1}, h^y_t \right) \leq \hat{e}_t.
\]

Given \( e_{t+1} \), write \( C_t(x_t; e_{t+1}, h^y_t) \) for total costs of producing \( x_t \) when agents born at \( t - 1 \) are constrained to provide the following generation with at least \( e_{t+1} \) units of the homogeneous good, that is,

\[
C(x_t; e_{t+1}, h^y_t) = e^m_t + b(n_t) + n_t \phi \left( \frac{c^o_{t+1}}{n_t} + \hat{e}_{t+1}, h^y_t \right).
\]

which, for the special case in which \( \phi(\cdot) \) is homogeneous of degree one, can be written equivalently as

\[
C(x_t; e_{t+1}, h^y_t) = e^m_t + b(n_t) + \phi \left( e^o_{t+1} + n_te_{t+1}, n_t h^y_t \right).
\]

Since \( \phi(\cdot) \) is convex, the cost function \( C(\cdot; e_{t+1}, h^y_t) \) must be a convex function and, given \((\hat{e}_{t+1}, h^y_t)\), the set of triples \((e^m_t, e^o_{t+1}, n_t)\) satisfying (8) is a convex set. It follows that any vector \( \hat{e}_t \) satisfying the necessary conditions in (6) is indeed a solution of the optimization problem in (4) or (5), that is, \( x_t(\hat{e}_t, \hat{e}_{t+1}) \) is implicitly defined by first order conditions in (6).

Notice also that each indirect utility function \( W_{t-1} \) is strictly increasing in \( e_t \), strictly decreasing in \( e_{t+1} \), and continuously differentiable on the interior of its domain. Therefore the slope \( \mu(\hat{e}_t, \hat{e}_{t+1}) \) of the indifference curve passing through an arbitrary point \((\hat{e}_t, \hat{e}_{t+1})\) is well defined. The Envelope Theorem yields

\[
\mu(\hat{e}_t, \hat{e}_{t+1}) = -\frac{\partial W_{t-1}(\hat{e}_t, \hat{e}_{t+1})}{\partial e_{t+1}} = \frac{\lambda_t(\hat{e}_t, \hat{e}_{t+1})}{\lambda_{t+1}(\hat{e}_t, \hat{e}_{t+1})}.
\]

Note that for every efficient allocation \( \hat{a} \in A \) one must have \( U_{t-1}(\hat{a}) = u(\hat{e}_t) = W_{t-1}(\hat{e}_t, \hat{e}_{t+1}) \).
where \( \lambda_t(\hat{e}_t, \hat{e}_{t+1}) \) and \( \lambda_{t+1}(\hat{e}_t, \hat{e}_{t+1}) \) are the Kuhn-Tucker multipliers for which the first order conditions of the optimization problem (4) are satisfied. Taking this into account we have

\[
\lambda_t(\hat{e}_t, \hat{e}_{t+1}) = u'_1(\hat{x}_t),
\]

and

\[
\lambda_{t+1}(\hat{e}_t, \hat{e}_{t+1}) = -u'_2(\hat{x}_t)\hat{n}_t.
\]

Therefore,

\[
\mu(\hat{e}_t, \hat{e}_{t+1}) = \frac{\hat{R}_{t+1}}{\hat{n}_t}.
\]

Since \( W_{t-1}(\hat{e}_t, \hat{e}_{t+1}) \) gives us the maximum utility that an agent can obtain without affecting the utility obtained by the next generation with the allocation \( \hat{a} \), the issue of whether \( \hat{a} \) is dynamically efficient can be reduced to a question on whether or not there exists a sequence \( \{e_\tau\}_{\tau=0}^\infty \) that improves the indirect utility of all generations. Next we present a result, similar to Balasko and Shell (1980, Lemma 5.4), that will be useful to dismiss allocations that are statically efficient but not dynamically efficient. This is the case when transfers to older generations are possible along all periods, so that a higher value of the indirect function \( W \) can be achieved.

**Lemma 4** Let \( \hat{a} \in A \) be an allocation satisfying the necessary conditions (6) and suppose \( \hat{a} \) is inefficient. Then there exists an allocation \( \tilde{a} \in A \) and some period \( \tau \geq 0 \), such that, for each \( t \geq 0 \), satisfies

\[
\tilde{e}_t \leq \hat{e}_t \text{ for all } t \geq 0, \text{ and } \tilde{e}_t < \hat{e}_t \text{ for all } \tau \geq t.
\]

**Proof.** Let \( \hat{a} \in A \) be an inefficient allocation satisfying the conditions (6), and let \( \tilde{a} \) be an allocation that Pareto dominates the allocation \( \hat{a} \), that is, an allocation satisfying

\[
W_{t-1}(\tilde{e}_t, \tilde{e}_{t+1}) \geq W_{t-1}(\hat{e}_t, \hat{e}_{t+1}) \text{ and } W_{\tau-1}(\tilde{e}_\tau, \tilde{e}_{\tau+1}) > W_{\tau-1}(\hat{e}_\tau, \hat{e}_{\tau+1}) \text{ for some } \tau = t.
\]

To show \( \tilde{a} \) satisfies condition (9), observe first that \( \tilde{a} \) verifies \( \tilde{e}_0 \leq \hat{e}_0 \), where the last inequality must be strict if \( U_{-1}(\tilde{a}) > U_{-1}(\hat{a}) \). Taking into account that \( W_{-1}(\cdot) \) is strictly increasing in \( e_0 \) one obtains

\[
W_{-1}(\tilde{e}_0, \tilde{e}_1) \leq W_{-1}(\hat{e}_0, \hat{e}_1).
\]

Also, since \( W_{-1}(\cdot) \) is strictly decreasing in \( e_1 \) and the inequality \( W_{-1}(\tilde{e}_0, \tilde{e}_1) \geq W_{-1}(\hat{e}_0, \hat{e}_1) \) is satisfied one obtains \( \tilde{e}_1 \leq \hat{e}_1 \), where the last inequality must be strict if either \( W_{-1}(\tilde{e}_0, \tilde{e}_1) > W_{-1}(\hat{e}_0, \hat{e}_1) \) or \( \tilde{e}_0 < \hat{e}_0 \) is satisfied. Proceeding analogously, since \( W_0(\cdot) \) is strictly decreasing in \( \tilde{e}_2 \) and the inequality \( W_0(\tilde{e}_1, \tilde{e}_2) \geq W_0(\hat{e}_1, \hat{e}_2) \) must be satisfied one must have \( \tilde{e}_2 \leq \hat{e}_2 \) (with \( \tilde{e}_2 < \hat{e}_2 \) if either \( W_0(\tilde{e}_1, \tilde{e}_2) > W_0(\hat{e}_1, \hat{e}_2) \) or \( \tilde{e}_1 < \hat{e}_1 \) is satisfied). By applying the argument recursively one obtains

\[
\nabla \tilde{e}_t = \hat{e}_t - \tilde{e}_t \geq 0 \text{ for all } t \geq 0
\]

and

\[
\nabla \hat{e}_t = \hat{e}_t - \hat{e}_t \geq 0 \text{ for some } \tau \text{ and all } t \geq \tau,
\]

which establishes condition (9) and, therefore, completes the proof of Lemma 4. \( \Box \)
3.3 Dynamic efficiency with endogenous fertility. The stationary case.

With the properties of the function $W_{t-1}$ in mind, the problem of dynamic efficiency is analyzed below. We will first provide some intuitions by considering stationary allocations (that is, allocations such that $a_t = a$ for all $t \geq 0$) in an economy with no technological progress (that is, such that $h^y_t = h^y$ for all $t$). Note that in this case one has $W_{t-1} = W$ for all $t \geq 0$, that is, the indirect utility function is the same for all generations. Since for any such stationary allocations one has $e_t = e_{t+1} = e$ for all $t \geq 0$, the set of stationary allocations is represented by the line $e_t = e_{t+1}$ in $\mathbb{R}_+^2$.

Consider now the point $e'$ in Figure 1, corresponding to an allocation $a'$ satisfying the necessary conditions in (6), with $e'_t = e'_{t+1} = e'$. Note that for such allocation one has $R'_n = R'_{n'} < 1$. Clearly, such allocation $a'$ is not efficient since by reducing $e'$ towards the point $e^g$ in the figure, all agents are better off. By contrast, consider now a point like $e''$, corresponding to an allocation $a''$ for which $e''_t = e''_{t+1} = e''$. Apparently, it is possible to improve all agents by increasing $e''$ in the direction of $e^g$. However, achieving Pareto improvements by increasing $e''$ is impossible, because increasing $e''$ implies that agents born at time $t = -2$ must decrease their consumption and, hence, their utility. Thus, the allocation $a''$ cannot be dominated by any other stationary allocation.

![Figure 1: Dynamic stationary efficient allocations when $W$ is strictly quasiconcave.](image)

If the indirect utility function is strictly quasi-concave, the slope of any indifference curve that intersects the line $e_{t+1} = e_t = e$ decreases as $e$ increases, and it is given by $\frac{\partial W}{\partial n}$. Thus, if utility function is strictly quasi-concave, then all allocations $a'$ for which $e'_t = e'_{t+1} = e' > e^g$ (that is, all inefficient allocations) verify $\frac{R'_n}{R'_n} < 1$, while all allocations $a''$ for which $e''_t = e''_{t+1} = e'' < e^g$ (that is, all stationary allocations that are not dominated by any other stationary allocation) verify $\frac{R''_n}{R''_n} > 1$. The stationary allocation $a^g$ for which $e^g_t = e^g_{t+1} = e^g$ has been referred to as the golden rule allocation, since it maximizes the utility obtained by a representative agent among those feasible, stationary allocations.
3.3.1 The non-convexity problem

Although, given $e_{t+1}$, the function $C_t(\cdot, e_{t+1}, h_t^y)$ is convex, the function $C_t(\cdot, h_t^y)$ need not be convex. This fact was first noted by Deardoff (1976) that showed that the golden rule allocation might not exist due to this fact. The following example illustrates this case.

**Example 1** (Samuelson, 1975 and Deardorff, 1976.) Consider an economy described by the following features and parametrizations: parents do not derive welfare from the children they have, i.e., $u(c_{mt}, c_{mt+1}, n_t) = v(c_{mt}, c_{mt+1}) = (c_{mt})^{\beta} (c_{mt+1})^{1-\beta}$ with $0 < \beta < 1$; parents have no costs of rearing children, i.e., $b(n_t) = 0$; human capital do not qualify workers for production, so $h_t = H(0) = 1$ and $d_t = 0$; and there is no technical progress, that is, $h_t^y = 1$. Let $F(k_t, 1) = f(k_t) = k_t^\alpha$, with $0 < \alpha \leq 1$.

In this case the feasibility constraints in (4) reduce to $c_{mt} + n_t (e_{t+1} + c_{mt+1}^y) = e_t$, and first order conditions yield $\hat{c}_{mt} = \beta e_t$, $\hat{c}_{mt+1} = \alpha(1-\alpha)(1/\alpha)^{-1}(1-\beta)e_t e_{t+1}^{1/(\alpha)}$, $\hat{n}_t = (1 - \alpha)^{1/\alpha}(1 - \beta)e_t e_{t+1}^{-1/(\alpha)}$, and $\hat{k}_t = [e_{t+1}/(1 - \alpha)]^{1/\alpha}$. The resulting indirect utility function is $W(e_t, e_{t+1}) = Be_t e_{t+1}^\gamma$, with $B = \beta^\beta [\alpha(1-\alpha)(1/\alpha)^{-1}(1-\beta)]^{1-\beta}$ and $\gamma = ((1/\alpha) - 1)(1 - \beta)$. Thus, $W$ is strictly quasiconcave whenever $\gamma < 1$ and strictly quasiconvex if $\gamma > 1$. Since all indifference curves pass through the origin, an interior, stationary allocation $\hat{a}$ for which $R = n$ does not exist in these two cases.

If the indirect utility function is not quasiconcave, condition $R = n > 1$ does not guarantee that a statically efficient, stationary allocation $a$ is also dynamically efficient. This case is illustrated in Figure 2 bellow, in which a point $(\hat{e})$ verifying condition $R = n > 1$ does not correspond to an efficient allocation.

![Figure 2: An example of a stationary allocation $\hat{e}$ that is not efficient when $W$ is not strictly quasiconcave.](image)

3.3.2 Dynamic efficiency with endogenous fertility. **The main result.** Due to the non-convexity problem discussed above, the simple criteria guaranteeing dynamic efficiency proposed by Cass (1972) and Balasko and Shell (1980) cannot be applied. In this section we present our main result, that extends Balasko and Shell’s criterium as explained bellow. Given a pair
Proposition 5

Consider an allocation \( \hat{a} \in A \) satisfying the necessary conditions (6). If

\[
\lim_{T \to \infty} \left( \frac{\hat{e}_T}{\prod_{t=0}^{T} \pi_t(\hat{e}_t, \hat{e}_{t+1})} \right) = 0.
\]  

(11)

then \( \hat{a} \) is Millian efficient.

Proof. Consider an allocation \( \hat{a} \in A' \) satisfying conditions (6), and let \( \tau \) be arbitrary. Suppose now that \( \hat{a} \) is weakly dominated by an allocation \( \tilde{a} \), and let \( \tau \) be the first period for which \( W_{\tau-1}(\hat{e}_t, \hat{e}_{t+1}) > W_{\tau-1}(\tilde{e}_t, \tilde{e}_{t+1}) \) (and, hence, \( \hat{e}_t < \tilde{e}_t \)) is satisfied. Write \( \epsilon = \hat{e}_t - \tilde{e}_t \). Since \( \hat{a} \) satisfies condition (P.4.1.1), there must exist a sufficiently large \( T^* \) such that, for each \( T > T^* \) one has

\[
\frac{\hat{e}_T}{\prod_{t=0}^{T} \pi_t(\hat{e}_t, \hat{e}_{t+1})} < \epsilon = \hat{e}_\tau - \tilde{e}_\tau.
\]

Then use condition (9) in the statement of Lemma 4 and the definition of \( \pi_t(\hat{e}_t, \hat{e}_{t+1}) \) to obtain the chain of inequalities

\[
0 < (\hat{e}_\tau - \tilde{e}_\tau) = \epsilon \leq \frac{\hat{e}_{\tau+1} - \tilde{e}_{\tau+1}}{\pi_\tau(\hat{e}_\tau, \hat{e}_{\tau+1})} \leq \frac{\hat{e}_{\tau+2} - \tilde{e}_{\tau+2}}{\pi_\tau(\hat{e}_\tau, \hat{e}_{\tau+1}) \pi_{\tau+1}(\hat{e}_{\tau+1}, \hat{e}_{\tau+2})} \leq \ldots \leq \frac{\hat{e}_{\tau+T} - \tilde{e}_{\tau+T}}{\prod_{t=\tau}^{\tau+T} \pi_t(\hat{e}_t, \hat{e}_{t+1})} \leq \frac{\hat{e}_T}{\prod_{t=\tau}^{T} \pi_t(\hat{e}_t, \hat{e}_{t+1})} \leq \frac{\hat{\pi}_T}{\prod_{t=\tau}^{T} \pi_t(\hat{e}_t, \hat{e}_{t+1})},
\]

which contradicts condition (11) and, therefore, establishes that \( \hat{a} \) is Millian efficient. \( \square \)
Remark 1. Apparently, the sufficient condition in (11) is not very useful, since one needs to compute all terms in the sequence \( \{\pi_t(\hat{e}_t, \hat{c}_{t+1})\}_{t=0}^{\infty} \). However, observe that if the sequence \( \{e_t\}_{t=0}^{\infty} \) converges to a steady state \( \hat{e} \), then condition (11) reduces to
\[
\pi(\hat{e}, \hat{c}) > 1;
\]
which, if the utility function \( W \) is quasiconcave reduces to
\[
\frac{\hat{R}}{\hat{n}} > 1.
\]

Remark 2. Observe that if the sequence \( \{e_t\} \) converges to a balanced growth path \( \hat{e}_t = \hat{e}_0 \gamma^t \) with constant \( \hat{R} \) and \( \hat{n} \), then condition (11) reduces to
\[
\pi(\hat{e}, \hat{c}) > 1 \text{ or } \hat{R} > \hat{n}\gamma.
\]

Remark 3. Observe that our condition (11) is weaker than those provided by Balasko and Shell (1980, Prop. 5.3). In their line, alternatively to (11), we could make two assumptions to guarantee the same result: [1] There exists a sequence \( \{\gamma_t\} \) such that the sequence \( \{e_t/\gamma_t\} \) is upper bounded; and, [2] \( \lim_{T \rightarrow \infty} \prod_{t=\tau}^{T} \frac{\gamma_t}{\pi_t(e_t, e_{t+1})} = 0 \).

To conclude the section, we present an example that illustrates how to apply the sufficient condition provided in this section to determine whether a given allocation is efficient or not. The example refers to affine technologies and shows that Balasko and Shell’s criterion might erroneously regard dynamically inefficient allocations as being efficient. To the extent that such affine technologies can be seen as reduced form representations of technologies arising in small open economies, such as those studied by Cigno (2003) or Grozen et al (2003), the example suggests that conclusions at which these authors arrive might be incorrect.

Example 2. Affine technologies and small open economies. As in example 1, suppose that \( F(k_t, h_t) = A_0 h_t + A_1 k_t \), with \( A_0 \geq 0, A_1 > 0 \) and that \( h_t = H(0, h^0) = 1 \) are satisfied.

In this case one has \( n_t k_{t+1} = \frac{n_t e_{t+1} + c_{t+1} - A_0 m_t}{A_1} \), and therefore the feasibility constraints in (4) reduce to
\[
e_t^m + n_t \left( b + \frac{e_{t+1} - A_0}{A_1} \right) + \frac{c_{t+1}}{A_1} = e_t,
\]
which, if the utility function takes the Cobb-Douglas form \( u(x) = e^m e^n x^{1/3} \), yields \( \hat{c}_t^m = \frac{1}{3} \hat{c}_t \), \( \hat{c}_{t+1} = \frac{4}{3} \hat{c}_t \), and \( \hat{n}_t = \frac{4}{3(A_1 b + (e_{t+1} - A_0))} \hat{c}_t \). Therefore \( W(\hat{c}_t, \hat{c}_{t+1}) = \frac{A_0^2 \hat{c}_t^3}{27w (A_1 b + (e_{t+1} - A_0))} \), which implies that \( W \) is in fact a strictly quasiconvex function, whose indifference curves, given by
\[
\left\{ e_{t+1} = \frac{A_0^2 \hat{c}_t^3}{27w} + (A_0 - A_1 b) : w > 0 \right\},
\]
are concave to the origin.

In order to analyze efficiency of stationary allocations, it is useful to distinguish between two possible cases, depending on the behavior of the term \( A_0 - A_1 b \). If \( A_0 - A_1 b < 0 \) is satisfied, then all indifference curves cross the \( e_t \) axis and for every point \( (e_t, e_{t+1}) = (e, e) \) corresponding to a stationary allocation one has \( \pi(e, e) > 1 \), which implies that every such stationary allocation is efficient. If, on the contrary, \( A_0 - A_1 b > 0 \) is satisfied, then all indifference curves cross the \( e_{t+1} \) axis at \( (e_t, e_{t+1}) = (0, 27 (A_0 - A_1 b)) \) and therefore \( \pi(e, e) < 1 \) for every stationary allocation \( a \), which implies that every such stationary allocation is inefficient.
A technology of the type described above can be seen as a reduced form representation of the type of technologies corresponding to a small open economy, as those studied, among others, by Cigno (2003) or Groezen et al (2003). These authors focus on models in which agents have the opportunity of lending or borrowing the only consumption good in the economy in an external financial market, and therefore face a resource constraint given by

$$\frac{c^0}{n_{t-1}} + c^m_t + b(n_t) + n_t \left( k^f_{t+1} - d^e_{t+1} \right) = g(k^f_t) - R_t d^f_t,$$

where $k^f_{t+1}$ represents per worker investment in physical capital and per worker and $d^e_{t+1}$ represents per worker debt.

Such technology can be represented equivalently as follows. Given $k_t \in \mathbb{R}$, let $k^f_{t+1}(k_t)$, $d^e_{t+1}(k_t)$ and $f(k_t)$ be defined by

$$f(k_t) = g(k^f_t(k_t)) - R_t d^f_t(k_t) = \max_{k^f_{t+1} \geq 0, d^e_{t+1} \in \mathbb{R}} \left\{ g(k^f_t(k_t)) - R_t d^f_t(k_t) : k^f_{t+1} - d^e_{t+1} = k \right\}.$$

With this notation, the resource constraint can be equivalently represented by $k_{t+1} = k^f_{t+1}(k_t)$, $d^e_{t+1} = d^e_{t+1}(k_t)$ and

$$\frac{c^0}{n_{t-1}} + c^m_t + b(n_t) + n_t k_{t+1} = f(k_t),$$

where $f(k_t)$ is defined as above. By writing $k(R_t)$ for the amount of physical verifying $g'(k(R_t)) = R_t$ one has

$$f(k_t) = g(k(R_t)) - R_t k_t = A_0(R_t) + R_t k_t.$$

Thus, the indirect utility of generation $t - 1$ is given by

$$W_{t-1}(\hat{e}_t, \hat{e}_{t+1}) = \frac{R_t^2 \hat{e}_t^3}{27 (R_{t+1} b + \epsilon_{t+1} - A_0(R_{t+1}))},$$

which is a strictly quasiconvex function. If the external return to investments converges to a stationary return $R^*$, one can proceed as explained above to determine whether a stationary allocation is efficient. \qed

4 A CHARACTERIZATION OF \( \mathcal{M} \)-EFFICIENT ALLOCATION AS DECENTRALIZED EQUILIBRIA

In this section, we show that every allocation satisfying the necessary conditions in Corollary 3 can be characterized as the outcome of a decentralized sequential price mechanism. This mechanism can be described as follows. At each date $t \geq 0$, two markets (to which all agents have free access) are open: a financial market, that allows agents to lend (or borrow) one unit of the homogeneous good in period $t$ and obtain (or pay back) $R_{t+1} \equiv (1 + r_{t+1})$ units of the same good in period $t + 1$, and a spot job market, in which efficient units of labor are exchanged against the homogeneous good at a price $w_t > 0$. In addition to these external markets, there exists a sequence of intergenerational contracts, represented by a sequence of fees $p_t \equiv \{p^f_{t+1} \}_{t \geq 0}$ that obliges all agents born at date $t$ to pay to their parents a fee equal to $p^f_{t+1}$ units of the homogeneous good when they reach their mature age. By paying this fee, the agents compensate their parents for the costs in which they incurred in order to make it possible that these agents were born. With these three markets operating at each date, price taking agents with perfect
foresight on future prices will behave as described below. Observe first that every agent, at the time he is born, needs to obtain funds to finance his education. Any children born at $t$ can get $d_t$ units of the homogeneous good at the interest rate $R_{t+1}$ in order to invest on his own education $h_{t+1}$; this allows him to obtain a total income $c_{t+1}$ (net of compulsory intergenerational transfers $p^e_{t+1}$) when he is a middle-age agent at $t+1$, that is,

$$e_{t+1} = w_{t+1}h_{t+1} - R_{t+1}d_t - p^e_{t+1}.$$  

Therefore, every agent will select the amount of resources to be invested when children to solve

$$\max_{d_t \geq 0} w_{t+1}H(d_t, h^y_{t}) - R_{t+1}d_t,$$

which yields an interior solution $d^*_t$ defined implicitly by

$$w_{t+1}H'_1(d^*_t, h^y_{t}) = R_{t+1}.$$  

A young adult with income $e^*_t$ will select a number of children $\hat{n}_t$ and an amount of savings $\hat{s}_t$ in the financial market to maximize his utility, that depends on the number of children he decides to rear and the consumption stream $(c^m_t, c^o_{t+1})$. The budget constraints faced by young adults will be given by

$$c^m_t + b(n_t) + s_t = e^*_t$$

and

$$c^o_{t+1} = n_t p^e_{t+1} + s_t R_{t+1}.$$  

Thus, an agent born at $t-1$ with income given by $I^*_t$ will select $x^*_t = (c^m_t, c^o_{t+1}, \hat{n}_t)$ and $s_t$ to solve

$$\max_{(x_t, s_t)} \{ u(x_t) : c^m_t + b(n_t) + s_t = e^*_t; c^o_{t+1} = n_t p^e_{t+1} + s_t R_{t+1} \},$$

which yields an interior solution $(x^*_t, s^*_t)$ defined implicitly by the above constraints and the equalities

$$\frac{u'_1(x^*_t)}{u'_2(x^*_t)} = R_{t+1} = \frac{p^e_{t+1}}{b'(n^*_t) - \frac{u_2(x^*_t)}{u'_1(x^*_t)}}$$

whenever $p^e_{t+1} \neq 0$

or

$$\frac{u'_1(x^*_t)}{u'_2(x^*_t)} = R_{t+1} \text{ and } b'(n^*_t) = \frac{u'_1(x^*_t)}{u'_2(x^*_t)}$$

whenever $p^e_{t+1} = 0$.

Finally, firms will select both their human and physical capital to maximize profits. In per worker terms, profits obtained in period $t+1$ by a firm using $(k_{t+1}, h_{t+1})$ as inputs will be given by

$$\pi_{t+1} = F(k_{t+1}, h_{t+1}) - w_{t+1}h_{t+1} - R_{t+1}k_{t+1}$$

Thus, firms will select $(k^*_t, h^*_t)$ to solve

$$\max_{(k_t, h_t)} F(k_{t+1}, h_{t+1}) - w_{t+1}h_{t+1} - R_{t+1}k_{t+1},$$

which yields an interior solution $(k^*_t, h^*_t)$ defined implicitly by

$$F_1(k^*_t, h^*_t) = R_{t+1}$$

and

$$F_2(k^*_t, h^*_t) = w_{t+1}.$$  

The notion of a decentralized equilibrium can be formalized as follows.

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Definition 6 For a given sequence of intergenerational contracts \( \{p_t^c\}_{t=0}^{\infty} \), an allocation \( a \in \mathcal{A}^t \) is a decentralized equilibrium generated by \( p^c \) if there exists a sequence of prices \( \{w_t, R_t\}_{t=0}^{\infty} \) such that for each \( t \geq 0 \):

\( i \) ) \( d_t \) solves (14)

\( ii \) ) \( (x_t, s_t) \) solves (15).

\( iii \) ) \( (k_t, h_t) \) solves (16) and (??).

Thus, an interior decentralized equilibrium associated to a sequence of fees can be equivalently characterized as an interior allocation \( a^* \) for which there exists a sequence of prices \( \{w_t^*, R_t\}_{t=0}^{\infty} \) such that, for each \( t \), the following conditions are satisfied:

\[
\begin{align*}
  w_{t+1}^* H_{t+1}^t (d_{t+1}^*, h_{t+1}^*) &= R_{t+1}^*; \\
  \frac{u_1'(x_t^*)}{u_2'(x_t^*)} &= F(t+1) \\
  \frac{u_1'(x_t^*)}{u_2'(x_t^*)} &= \frac{p_{t+1}^c}{b'(n_t^*) - \frac{u_1'(x_t^*)}{u_1'(x_t^*)}} \quad \text{if } p_{t+1}^c \neq 0; \quad \text{and} \\
  b'(n_t^*) &= \frac{u_2'(x_t^*)}{u_1'(x_t^*)} \quad \text{if } p_{t+1}^c = 0; \\
  F_1(k_{t+1}^*, h_{t+1}^*) &= R_{t+1}^* \\
  F_2(k_{t+1}^*, h_{t+1}^*) &= w_{t+1}^*;
\end{align*}
\]

and

\[
c_t^0 + n_{t-1} \left[ e_t^m + b(n_t) + n_t (d_t + k_{t+1}) \right] = n_{t-1} F(k_t, h_t) \\
  n_t (d_t + k_{t+1}) = s_t.
\]

These equations yield a straightforward characterization of statically efficient allocations as equilibria of the sequential price mechanism described above, as the following result states.

Theorem 7

(i) Consider an arbitrary sequence \( p^c \) of intergenerational contracts, and let \( a^* \) be an interior decentralized equilibrium generated by \( p^c \). Then \( a^* \) is statically \( \mathcal{M} \)-efficient.

(ii) For every statically \( \mathcal{M} \)-efficient interior allocation \( a^* \), there exists a sequence \( p^c \) of intergenerational contracts generating \( a^* \) as a decentralized equilibrium.

Proof. To prove (i), let \( a^* \) be an interior decentralized equilibrium generated by a sequence \( p^c \), and let \( \{R_t^*, w_t^*\}_{t=0}^{\infty} \) be the sequence of wages and interest rates corresponding to such equilibrium. Using (17), it is straightforward to check (17) that \( a^* \) satisfies, for each \( t \geq 0 \),

\[
R_{t+1}^* = \frac{u_1'(x_t^*)}{u_2'(x_t^*)} = F_2(k_t^*, h_t^*) = F_2(k_t^*, h_t^*) H_{t+1}^t (d_t^*, h_t^*).
\]
Thus, in order to show that \( a^* \) satisfies the necessary conditions in (6) (and, hence, it is statically efficient), we simply need to show that

\[
R_t^{*+1} = \frac{c^*_t / n_t^*}{b'(n_t^*) - \frac{w_i^*(x_t^*)}{u_i^*(x_t^*)} + (k_t^* + d_t^*)}
\]

is satisfied. To prove the last equality (18) holds, use the market clearing condition in capital markets and the equality \( b'(n_t^*) = \frac{w_i^*(x_t^*)}{u_i^*(x_t^*)} \) \( R_t+1 = p_t^* \) in (17) to write the term at the right hand side of (18) as

\[
\frac{c^*_t / n_t^*}{b'(n_t^*) - \frac{w_i^*(x_t^*)}{u_i^*(x_t^*)} + (k_t^* + d_t^*)} = \frac{p_t^* + \frac{s_t^*}{n_t^*}R_t^*}{p_t^{*+1} + \frac{s_t^*}{R_t^{*+1}} + \frac{s_t^*}{n_t^*}} = R_t^{*+1},
\]

which establishes (i).

To prove(ii), let \( a^* \) be an interior, statically efficient allocation and \( R_t^{*+1} \) be the rate of return implicitly defined by the necessary conditions in (6). Then using the equilibrium conditions in (17), it is straightforward to check that a sequence of intergenerational transfers \( \{p_t^{*+1}\} \) defined by \( b'(n_t^*) - \frac{w_i^*(x_t^*)}{u_i^*(x_t^*)} \) \( R_t+1 = p_t^* \) generates \( a^* \) as an interior, decentralized equilibrium, which completes the proof of the (ii) statement in Theorem 7.\( \square \)

Thus, the notion of Millian efficiency admits a characterization that is closely analogous to the one provided by the two fundamental Theorems of Welfare Economics. Nevertheless, two important differences arise.

With respect to the (i) statement in Theorem 7, that can be regarded as a version of the First Fundamental Welfare Theorem, it is important to observe that the equilibrium generated by a sequence of contracts for which \( p_t^* = 0 \) for all \( t \) is statically \( \mathcal{M} \)-efficient. Although other authors have reached to similar conclusions, such as Nerlove, Razin and Sadka (1985) in a two-period framework or Groezen et al (2003) in an infinite-period setting, they both identify an efficient allocation with a maximum of a Millian social welfare function. In the case of Groezen et al (2003), their version of the (i) statement in Theorem 7 applies only to a parametric example of a stationary economy, and it is required that a given parameter determining the agents’ preferences equals the (constant) intergenerational discount factor in the definition of the Millian social welfare function. Therefore their result is not robust to changes in the specifications of the agents’ preferences.

Both Nerlove, Razin and Sadka (1985) and Groezen et al (2003) interpret such equilibria as the natural, laissez faire equilibrium that would arise in an economy with free access to capital markets. However, we do not entirely agree with such interpretation. First, because children cannot in general sign binding contracts, which makes it hard that they can find resources for human capital accumulation, as argued, among others, by Boldrin and Montes (2003). Second, if they could sustain contracts to finance human capital accumulation as implicit contracts inside the family, as argued by Erhlich and Lui (1991), Conde-Ruiz et al (2003) and Cigno (1993, 2000, 2003), then sustaining such implicit contracts as subgame perfect equilibria of an implicit contracting game imposes that the intergenerational transfers from parents to children are strictly lower than those needed to finance capital accumulation (which in our setting would correspond to contracts for which \( p_t^* > 0 \)). In view of this, we might alternatively consider the equilibria generated by \( p_t^* \equiv 0 \) as an equilibrium with missing markets, since there is no market in which parents and their offspring can bargain on the right to exist. In this sense, this version of the First Theorem of Welfare Economics contrasts with well known results on welfare properties of economies with missing markets. Here, the absence of a market yields no efficiency loss, at least if one regards efficiency as Millian efficiency.
The (ii) statement of Theorem 7 can be regarded as a version of the Second Fundamental Theorem of Welfare Economics. Similar to the case of economies with exogenous population, every Millian efficient can be decentralized by initially selecting an appropriate sequence of intergenerational transfers, and then allowing the agents to determine their consumption and investment decisions at competitive markets. Differently from the standard, exogenous population case, an incentive scheme that links intergenerational transfers with fertility decisions is needed. More precisely, for every system of intergenerational transfers that achieves Millian efficiency, every young adult has to pay a lump-sum tax $p_1^t$ (or, in some cases, receive a lump-sum subsidy given by $-p_1^t$), and every old adult will receive a subsidy (or pay a tax) which depends linearly of the number of children she decides to have. Thus, our version of the Second Theorem of Welfare Economics provides a justification of social security programs with child allowances, as pointed out by Groezen et al (2003).

We should also point out that the characterization given above refers to statically $\mathcal{M}$-efficient allocations. Of course, many of the equilibria described above might be dynamically inefficient, and therefore it is worth exploring what type of intergenerational contracts ensure that dynamic efficiency is achieved.

5 Strong efficiency

In this section, we discuss an alternative notion of Pareto efficiency that strengthens the Millian notion, which we refer to as strong efficiency. It differs from the Millian notion explored in previous sections in two respects. First, it is based on a notion of Pareto dominance that compares not only symmetric allocations, but also asymmetric ones. Second, it takes into account for social welfare judgements all potential agents in the economy, not only those who get to be born.

To simplify things, we first focus on the particular class of asymmetric allocations, which we denote by $\mathcal{A}_T$, where new individuals, and their descendants, appear in the economy from some given period $T > 0$ on. For any allocation in this class, there are two types of agents (indexed by $\delta = \{1, T\}$), that might be born in every period $t$. Agents born at period -2 and -1 are assigned to type 1. Individuals of the first type live through every period $t \geq 0$, agents of the second type are not born until date $T \geq 0$ is reached. In period $T$, both types of agents are born from agents of the first type, and in successive periods $t > T$ all agents of a given type are born from agents of the same type.

At each date $t \geq 0$, and for each $\delta = \{1, T\}$, a decision $a_t^\delta = (c_t^{m\delta}, c_{t+1}^{\delta}, n_t^\delta, k_{t+1}^\delta, d_t^\delta)$ is taken. Taking into account that, before $T$ is reached, only agents of the first type are born, the feasibility constraints that an allocation $a_t^T = \{a_t^\delta = (c_t^{m\delta}, c_{t+1}^{\delta}, n_t^\delta, k_{t+1}^\delta, d_t^\delta)\}_{\delta \geq 0; \delta = \{1, T\}} \in \mathcal{A}_T^T$ must satisfy are given by:

\[
(n_{-2}^1, n_{-1}^1, k_0^1, h_0^1) = (1, \pi_{-1}, \bar{h}_0, \bar{h}_0),
\]

\[
c_t^{o1} + n_{t-1}^1 [c_t^{m1} + b(n_t^1 + n_t^1 (d_t^1 + k_{t+1}^1))] \leq n_{t-1}^1 F^1(h_t^1, k_t^1),
\]

and

\[
a_t^T = (c_t^{mT}, c_{t+1}^T, n_t^T, k_{t+1}^T, d_t^T) = (0, 0, 0, 0, 0),
\]

whenever $0 \leq t < T$;

\[
c_t^{o1} + n_{t-1}^1 [c_t^{m1} + b(n_t^1 + n_t^T) + n_t^1 (d_t^1 + k_{t+1}^1) + n_t^T (d_t^T + k_{t+1}^T)] \leq n_{t-1}^1 F^1(h_t^1, k_t^1),
\]
whenever \( t = T \); and
\[
\sum_{\delta = \{1, T\}} N_{t-1}^\delta \left\{ c_t^\delta + n_t^\delta \left[ c_{t+1}^\delta + b(n_t^\delta) + n_t^\delta \left( d_t^\delta + k_{t+1}^\delta \right) \right] \right\} \leq \sum_{\delta = \{1, T\}} N_{t-1}^\delta F^\delta(h_t^\delta, k_t^\delta),
\]
whenever \( t > T \).\(^5\)

Let \( \hat{A} \equiv \bigcup_{T=0}^{\infty} \tilde{A}_T \), that is, \( \hat{A} \) is the set formed by all allocations in the class described above. Note that any symmetric allocation \( a \in A \) can be identified with an asymmetric allocation \( \tilde{a} \in \hat{A} \) such that \((c_t^\delta, c_{t+1}^\delta, n_t^\delta, k_t^\delta, d_t^\delta) = 0 \) for all \( T \). In what follows, we will assume that agents of the same type undertake the same decisions; that is, we treat asymmetrically agents born and not born, keeping the assumption of symmetry among those born, so that \( A \subset \hat{A} \).

Assume that, given an allocation \( a \in \hat{A}_T \), preferences of an agent of type \( \delta \) born at date \( t - 1 \) will be represented by a utility function
\[
U_{t-1}^\delta(a) = \begin{cases} 
0 & \text{whenever } t = -1 \\
u(c_t^\delta, c_{t+1}^\delta, n_t^\delta + n_t^T), & \text{whenever } t = T \\
u(c_t^\delta, c_{t+1}^\delta, n_t^\delta), & \text{otherwise.}
\end{cases}
\]

With this notation, we now proceed to propose a welfare criterion that allows one to compare any two allocations in \( \hat{A} \). Although extending the Millian criterium explored in previous sections is straightforward, we will take a step further and provide a criterium that takes into account all potential agents in the economy, not only those who get to be born.

With this objective in mind, we assume that there exists a utility threshold, \( \overline{u}_{t-1} \), above which any agent of any type \( \delta \) is better off by being born at \( t - 1 \) and obtain utility \( U_{t-1}^\delta(a) \geq \overline{u}_t \) than he would be if he was not born. One might argue that determining this threshold constitutes an extremely difficult task, mainly because agents directly affected by fertility decisions are not able to answer the sort of questions that are needed to determine this threshold (that is, questions of the kind “would you like to be born and obtain utility above \( \overline{u}_t \) or you would rather prefer not to be born?”). However, this difficulty does not mean that society should not afford these sort of questions. In fact, we shall argue below that the notion of Millian efficiency explored in previous sections implicitly determines a sequence of thresholds \( \{\overline{u}_t\} \) of the form described above.

Once a decision concerning to utility derived by agents who never get to be born is made is taken, extending the Pareto criterium to the set \( A^* \) is straightforward.

**Definition 8** A feasible allocation \( \tilde{a} \in A^* \) dominates an allocation \( a \in \hat{A} \) if, for each \( T \geq 0 \) and each \( \delta = \{0, 1, 2, ...T\} \) one has
\[
U_{t-1}^\delta(\tilde{a}) \geq U_{t-1}^\delta(a), \quad \tilde{n}_t^\delta \geq n_t^\delta \text{ whenever } U_{t}^\delta(\tilde{a}) \geq \overline{u}_t, \quad \tilde{n}_t^\delta \leq n_t^\delta \text{ whenever } U_{t}^\delta(\tilde{a}) \leq \overline{u}_t;
\]
with some of the inequalities above being strict for at least one \( \tau \geq 0 \) or some \( \delta = \{1, 2, ...T, \ldots\} \).

An allocation \( \tilde{a} \in \hat{A} \) dominates an allocation \( a \in \hat{A} \) if i) it gives more utility to all agents who are born under the two allocations; and, ii) it allows more (respectively, fewer) people to be alive whenever they obtain higher (respectively, lower) utility than the threshold utility level \( \overline{u}_t \).

\(^5\)Note that, although we are implicitly assuming that there are two production units (one for each type of agents) operating at time \( t \geq \tau \), this assumption is without loss of generality, provided \( F \) exhibits constant returns to scale.

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An allocation $a \in \tilde{A}$ is said to be strongly efficient (or simply, $S$-efficient) in $\tilde{A}$ if it is not Pareto dominated by any other allocation $a \in \tilde{A}$.

It is straightforward to see that this notion of $S$-efficiency is indeed stronger than the Millian one. We regard the notion of strong efficiency as an adaptation of the notion of Golosov et al (2003)’s $P$-efficiency to a set of potential agents is a continuum. Similar to their notion, it is based on a notion of Pareto dominance that takes into account all potential agents in the economy (which are assumed to derive a given utility of non-being alive, which in their case is normalized to be given by $\pi = u(0)$) and it allows one to compare symmetric allocations with asymmetric ones (although, in our case, only a special type of asymmetry is allowed). In their discrete type setting, considering non-symmetric allocation brings no problems, as it would in ours.

In spite of this, taking into account this type of asymmetries reduces drastically the set of Millian efficient allocations that are also strongly-efficient. To see this, it is useful to restrict ourselves to the set of stationary, symmetric, Millian efficient allocations in an economy with no growth. Without loss of generality, suppose the allocation $a$ corresponding to a constant fee $p^e$ yields an indirect utility $W(e, e) > \pi$. In this case, it is straightforward to show that $a$ is dominated by many allocations in $\tilde{A}^*$. A particular allocation $\tilde{a} = (\tilde{a}^1, \tilde{a}^2) \in \tilde{A}_2$ can be constructed as follows.

First, let $\tilde{a}^1$ be such that $\tilde{a}^1_t = a$ for every period $t > 2$, and let $\tilde{a}^2$ be such that $\tilde{a}^2_t = a_t$ for $t > 2$. The only component of $\tilde{a}$ that remains to be specified is $\tilde{a}^2_2$. Choose now $\tilde{a}^2_2$ in such a way that $\tilde{a}^2_2 > 0$, $\tilde{a}^2_2 < e$ and $W(\tilde{e}^2_2, e) > \pi$. Observe that such number exists provided $W$ is continuous. Clearly, the allocation $\tilde{a}$ is feasible and provides all agents with higher utility, which establishes that $a$ is not strongly efficient.

The argument can be extended to any allocation $a$ for which $W(e, e) > \pi$ and, since choosing $e_t = 0$ is always feasible, a similar argument holds for allocations for which $W(e, e) < \pi$. The set of stationary Millian efficient allocations that are also $S$-efficient is reduced therefore to those satisfying

$$W(e, e) = \pi.$$

This conclusion seems obviously undesirable and puts some obvious doubts on the notion of strong efficiency, specially if it is also assumed that $\pi = u(0)$, which leaves the allocation $\tilde{a} = 0$ as the only symmetric $S$-efficient allocation. Nevertheless, some remarks on this conclusion are in order.

First, the conclusion depends crucially on the fact that we are analyzing the strong-efficiency of symmetric allocations. However, if strong efficiency is analyzed for asymmetric allocations and only a small measure of agents obtain the utility threshold in every period, the argument cannot be extended. The conclusion would not be the same either if no children can exceed a threshold.

Second, the notion of $S$-efficiency might be deemed undesirable on other grounds, independently of this conclusion. The notion of Pareto dominance underlying the notion of strong-efficiency implies that agents with identical characteristics can be treated asymmetrically. Put it in terms of contracts sustaining Millian efficient allocations, in order to achieve strong efficiency, we should allow parents to behave as discriminating monopolists with their own children. Whether this notion represents the Social Contract for some society, we may wonder about other economic roles that children may play in such society for being treated different by their parents.

Third, even though the conclusion affecting stationary strongly efficient allocations seems undesirable, exploiting certain Pareto improvements (at a certain cost in terms of symmetry) might be not so undesirable. In the example given above, only the second type of agents born at $t$ are treated asymmetrically, and infinitely many more agents are allowed to be alive. Moreover, symmetry might be restored by selecting the discriminated individuals by a random device or by discriminating agents that are not equal. This type of discrimination, however, might have a cost
in terms of Millian efficiency.

Fourth, there are other ways to extend the Pareto criterium that might be worth exploring. One possible way to achieve this would be by defining a Pareto order that takes into account all the potential agents in the economy but still restricts the set of Pareto comparable allocations to be formed by the set of symmetric allocations. This would imply that a restriction of the form $(n_t - \bar{n}_t) (u(\bar{x}_{t+1}) - \bar{u}) \geq 0$ should be added into the definition of the indirect utility function. Another possibility would arise by distinguishing between an upper threshold $U_{t-1}$ and a lower threshold $\bar{U}$ in such a way that agents obtaining $U_{t-1} > \bar{U}$ are assumed to be better off by being alive, while agents obtaining $U_{t-1} < \bar{U}$ are assumed to be worse off by being alive.

6 Conclusions

This paper studies the issue of Pareto efficiency in an overlapping generations setting with endogenous population. In an environment in which the set of agents is endogenous, we adopt a weak extension of the notion of Pareto efficiency, referred to as Millian efficiency. The notion of Pareto dominance underlying this notion of Millian efficiency is based exclusively on preferences of those agents alive, and allows only for paralleling symmetric allocations (i.e., allocations in which all living individuals of the same generations take the same decisions). We provide necessary conditions that every Millian efficient allocation must satisfy, and a sufficient condition determining whether a given allocation satisfying these necessary conditions is Millian efficient. With these results at hand, we characterize Millian efficient allocations as the equilibria of a decentralized price mechanism. Finally, we discuss an alternative extension of the Pareto criterium that strengthens the Millian notion, referred to as strong efficiency.

This theoretical study of efficiency with endogenous population has relevant implications for analyzing the role of social institutions, such as the welfare state programs, in achieving efficiency. We provide a setting for discussing on welfare grounds any proposal of rethinking the role of the State in modern societies (see Becker and Murphy, 1988) and consequently a redesign of institutions that may affect fertility decisions on efficiency basis, extending works that assumed exogenous population growth like Boldrin and Montes (2002).

References


6.1 Proof of Proposition 4.

Let \( \tilde{a} \in A \) be an inefficient allocation satisfying the conditions (6), and let \( \tilde{a} \) be an allocation that Pareto dominates the allocation \( \tilde{a} \). Then by definition of efficiency it is clear that the following statement holds:

\[
W_{t-1}(\tilde{e}_t, \tilde{e}_{t+1}) \geq W_{t-1}(\tilde{e}_t, \tilde{e}_{t+1}) \quad \text{and} \quad W_{\tau-1}(\tilde{e}_\tau, \tilde{e}_{\tau+1}) > W_{\tau-1}(\tilde{e}_\tau, \tilde{e}_{\tau+1}) \quad \text{for some} \quad t = \tau.
\]

To show \( \tilde{a} \) satisfies condition (9), observe first that \( \tilde{a} \) verifies

\[
\tilde{e}_0 \leq \hat{e}_0,
\]

where the last inequality must be strict if \( U_{-1}(\tilde{a}) > U_{-1}(\hat{a}) \). Taking into account that \( W_{-1}(\cdot) \) is strictly increasing in \( e_0 \) one has

\[
W_{-1}(\tilde{e}_0, \hat{e}_1) \leq W_{-1}(\hat{e}_0, \hat{e}_1).
\]

Also, since \( W_{-1}(\cdot) \) is strictly decreasing in \( \hat{e}_1 \) and the inequality \( W_{-1}(\tilde{e}_0, \hat{e}_1) \geq W_{-1}(\tilde{e}_0, \tilde{e}_1) \) must be satisfied one must have

\[
\tilde{e}_1 \leq \hat{e}_1,
\]

where the last inequality must be strict if either \( W_{-1}(\tilde{e}_0, \hat{e}_1) > W_{-1}(\tilde{e}_0, \tilde{e}_1) \) or \( \tilde{e}_0 < \hat{e}_0 \) is satisfied. Proceeding analogously, since \( W_0(\cdot) \) is strictly decreasing in \( \tilde{e}_2 \) and the inequality \( W_0(\tilde{e}_1, \tilde{e}_2) \geq W_0(\hat{e}_1, \tilde{e}_2) \) must be satisfied one must have

\[
\tilde{e}_2 \leq \hat{e}_2,
\]

where \( \tilde{e}_2 < \hat{e}_2 \) must be satisfied if either \( W_0(\tilde{e}_1, \tilde{e}_2) > W_0(\hat{e}_1, \tilde{e}_2) \) or \( \tilde{e}_1 < \hat{e}_1 \) is satisfied. By applying the argument recursively one obtains

\[
\nabla \tilde{e}_t = - (\tilde{e}_t - \hat{e}_t) \geq 0 \quad \text{for all} \quad t \geq 0
\]

and

\[
\nabla \tilde{e}_\tau = - (\tilde{e}_\tau - \hat{e}_\tau) \geq 0 \quad \text{for at least one period} \quad \tau
\]

which establishes condition (9) and, therefore, completes the proof of Lemma 4. \( \square \)

6.2 Proof of Theorem 7

The proof is straightforward. A little bit of algebra shows that conditions (17) are equivalent to the necessary conditions in (6). \( \square \)