Age distribution and age heterogeneity in economic profiles as sources of conflict between efficiency and equity in the Solow-Stigliz framework

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Summary

Much of the debate on the role played by the age distribution of the population in shaping the growth regimes of the neoclassical growth model of Solow has focused on the notion of capital dilution and the possibility that “optimal” population growth regimes (OPGR) exist. Conversely less has been done as regards the implications of growth regimes for distribution, probably because Stiglitz’s (1969) fundamental result predicts that economic growth sets in motion forces of an “essentially” egalitarian nature.

In this paper we start to investigate how distinct regimes of population growth and age distribution affect income and wealth distribution in a Solow-Stiglitz framework embedding a single but fundamental “inequality preserving” force, e.g. an heterogeneous age profile of labour productivity. Our main results are that 1) a population growth regime characterised by minimal inequality (a “MinIPGR) often exists, but 2) it is usually amazingly “far” from the corresponding OPGR, when also an OPGR exists. This seems to suggest a potentially striking trade-off between efficiency and equity during economic growth.

Key-words: neoclassical growth model, population growth and age distribution, income distribution, Optimal Population Growth Rate, Minimal Inequality Population Growth Rate

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Introduction

A long standing problem, emerged as a threat to the pessimism of Solow's capital dilution proposition, is that of the optimal population (e.g. Samuelson (1975, 1977), Deardorff (1976), Arthur-McNicoll (1978), Blanchet (1988)). In particular the possibility of the existence of an optimal population growth rate (OPGR) has been put forth, that is whether an "efficient" PGR does exist which could be approached by demographic policy interventions, or just by chance. The possibility of the existence of OPGR naturally emerges as soon as one takes the age structure of the population, a factor disregarded in the original Solow's model, into account.

It is a documented fact that the age of the labour force and of the overall population is a major source of heterogeneity in economic patterns (consumption and saving, productivity, work participation, etc). Arthur and Mc Nicoll (1978) have shown that the basic neoclassical growth model of Solow can yield much richer predictions if one properly use the information embedded in the population term. Indeed, the fact that the population/labour supply is exogenously growing at the constant rate n (as postulated by Solow's model and indeed by all economic models of exogenous growth), essentially means that the underlying population is "stably" growing with an unchanging age distribution. This allows to link the neoclassical model with the equilibrium theory of age structured population dynamics. A major achievement of this link is that we are thus enabled to take into account the effects of virtually every type of age heterogeneities at the desired level of detail.

Since Arthur and Mc Nicoll (1978) several issues related with the neoclassical growth model under a stable population dynamics have been investigated (intergenerational transfers, existence of optimum population growth rates (OPGR), consequences of several types of heterogeneity, etc). However little seems to have been done as regards the relation between growth and distribution in the neoclassical model in presence of age structure and age related heterogeneity.

The issue of wealth and income distribution within the Solovian model has been exhaustively investigated by Stiglitz (1969), who argued that under homogeneity of economic parameters (even in the presence of groups with different reproduction rates positively depending on their income), the wealth distribution in the long run tends to be equalitarian, though there are instances of forces which tend to make wealth unevenly distributed, as is the case of heterogeneity in the productivity of labour.

In this paper, we first review the literature on the Solow's model with population age structure, by discussing the fundamental notions of intergenerational transfer effect, and of, respectively, Optimal Population Growth Rate and Worst Population Growth Rate (WPGR, a concept not yet pointed out in the literature), and second, we study the relation between growth and distribution by focusing on a simple Solow-type model with population and Cobb Douglas production function which embeds a unique, but fundamental, heterogeneity in the productivity profile by age, which is a stylised fact of labour economics.

The model shows the presence, additionally to the well known capital dilution and intergenerational transfer effect, of a productivity effect which, besides affecting the location of the OPGR and WPGR of the economy, prevents the possibility of Stiglitz egalitarian result and seems to have a complex influence on the degree of inequality in the neoclassical model. Our results show: 1) the existence of values of the PGR's which can be inequality-minimiser or inequality maximiser, by introducing from this standpoint, a new role for the population growth rate in terms of equity, and 2) the fact that such values might be at all different by the OPGR and WPGR previously discovered. The latter result sheds a new light on the possible complicated trade-off between efficiency and equity emerging as a consequence of a given demographic policy. These results are, as far as we know, novel in the literature.

The present paper is organised as follows. Sections 2 shortly reviews the literature on age structured population dynamics within Solow's model. Section 3 illustrates the concepts of capital dilution and intergenerational transfer and shows examples of the existence of both OPGR and WPGRs. Section

4 introduces our Solow-type model with heterogeneity in the productivity profile by age, and section 5 investigates its consequences for income distribution according to Stiglitz's (1969) standpoint. Conclusive remarks follow.

2. Neoclassical growth models and population age structure: a review

As Lee (1994) pointed out two main frameworks have been used in the literature for studying the allocation of resources across age. On the one hand OLG models, extensively used since Samuelson's 1958 work, "pave the way for a deeper integration of demography and macroeconomics than has yet proven possible". OLG models allow rather deep dynamic analyses but suffer the drawback of necessarily relying on a very simplistic representation of the population life cycle that prevents some of the most basic questions to be properly posed. The second strand has a more demographic standpoint, and exploits the richness of the apparatus of the theory of linear age structured populations dynamics. This stands at the very core of demography, e.g. Lotka's and Leslie's stable population model (Keyfitz 1990). The fundamental result of this theory predicts that a closed population exposed to unchanging fertility and mortality rates eventually achieves an asymptotic regime of stable ("balanced") growth characterised by exponential growth of all aggregate variables (total population, births, and deaths) and by an unchanging age profile. Thus the theory is immediately prone to be combined for instance with descriptive models of economic growth with exogenous population. Among these the natural candidate to such amendments is certainly Solow's (1956) model for its "symmetry" (Arthur and McNicoll 1977) with the demographic problem (another candidate could be Goodwin's (1967) model, though the endogeneity of employment causes some extra difficulty). Most of the available contributions in this second strand of the literature deal indeed with Solow-type models with age structured demography. By passing, we recall that, as known, the textbook Solow's model, where the population age structure is not taken into account, predicts that the long term level of capital per worker is a monotonically decreasing function of the population growth rate (n). This "capital dilution" effect seems to suggest that countries characterised by the higher rates of population growth will, ceteris paribus, experience poorer long term conditions. Thus apparently not only slowering population growth is always desirable, but, in particular (provided a balanced growth state exists, e.g. $\delta+n>0$) sustained population decay seems to be the best growth condition.

The age structured extension of Solow's model, which is relatively easy in technical terms (population dynamics remains fully exogenous), provides a surprisingly rich body of new results, especially as far as equilibria are concerned. A major merit of this approach is that it allows to embed in the model virtually of types of age heterogeneities - as well known age is a major source of variability in economic patterns - with the desired level of detail, thus avoiding limitations of low dimension OLG models. The first contributors to this strand are Arthur and Mc Nicoll (1977) who studied the problem of optimal welfare dynamics (e.g. under a social planner) in a one-good Solow type economy with age structure of both the population and the capital stock. Arthur and Mc Nicoll (1978) have reconsidered Samuelson's (1975) problem in the context of a Solow's model with age structure. They have shown that compared to the standard Solow's model the pure fact of taking the age structure of the population into account allows the appearance of an "intergenerational transfer" effect that can in principle counterbalance Solow's capital dilution effect and make the impact of population change on welfare less easy to predict, e.g. dependent on actual economic circumstances (though in the end they conclude that in most cases the intergeneration transfer term will have the same sign of the capital dilution one, thereby reinforcing Solow's result). Lee (1980) has considered the impact on consumption profiles by age of changes in the growth rate of the population in age structured extensions of Samuelson (1958) and Solow (1956) models. He also gave clues on how to extend the treatment from individuals to households, which is the truly correct unit for this type of analyses. Blanchet (1988,1992) has investigated several issues related to the balance between capital dilution and intergenerational transfers. In particular he has shown the existence of OPGR in the simplest age structured Solow-type model. Additionally he has shown that Solow's model might overestimate capital dilution, in that significantly different results arise when the simplistic assumption of capital depreciation at constant rate is removed in favour of more realistic assumptions on the demography of capital. Lee (1994) has significantly expanded his previous framework for the investigation of intergenerational transfers under regimes of changing population dynamics. Several efforts have moreover been devoted to apply the aforementioned methodologies to evaluate the impact of recent trends of population change, such as changing age distributions due to the demographic transition in developing economies (McNicoll, 1985), and sustained population aging due to the progressive onset of below replacement fertility in western economies (Mc Nicoll, 1987). Cutler et al. (1990) have investigated the impact of population ageing by using measures of economic dependency. Weil (1997) reviews previous works on the economic impact of regimes of changing age distributions. Weil (1999) suggests that the major impact of changing age distributions on golden rule consumption could occur during transients, e.g. during the true ageing phase, and not in the long term "older" population. As regards income distribution Lam (1984,1997) has devoted many efforts to the investigation of the various ways in which demographic variables affect distribution. Lam (1989) has shown that the lifetime wage is minimised under a flat age distribution. On the more applied side, Lindh (1999) and Lindh-Malmberg (2000) have shown, by an econometric analysis of a Mankiw-Romer-Weil (1992)-type model with age structure of the population and heterogeneous labour productivity, that changing age distributions in OCDE countries during 1950-1990 had an important role in shaping the observed growth rates of the economy in the same period.

3. Capital dilution and intergenerational transfer effects: OPGR and WPGR in the neoclassical growth model

In the textbook Solow-type model an incorrect identification is often made (explicitly or implicitly) between "per-worker" and "per-head of population" (or per-capita) quantities. These quantities are by no means identical: the latter is equal to the former times the fraction of the active population that is actually employed. Under some simplifying assumptions (constant participation and employment rates across ages) the latter is given by the fraction of the population in the work age span, which in turn depends on the overall age distribution of the population.

Let us consider the equilibrium state of a standard Solow's model with exogenous population when the age structure of the population is taken into account. For simplicity we only consider the case of stable populations (this part reviews Blanchet 1988,1992). This amounts to assume that the population is stably evolving in a given growth regime specified by a couple *(n,SAD(n))*, where *n* is the growth rate (typically considered by Solow-type models), and *SAD(n)* denotes a stable age distribution of the population (disregarded in textbook Solow models). In this case the equilibrium of the model, under the usual constant returns to scale Cobb-Douglas production function in per-worker terms $y = Qk^{\alpha}$, $0 < \alpha < 1$, is characterised by the quantities

$$k^* = \left(\frac{\sigma A}{\delta + n}\right)^{\frac{1}{1-\alpha}} \qquad \delta + n > 0 \tag{3.1a}$$

$$y^* = A\left(k^*\right)^{\alpha} = A\left(\frac{\sigma A}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} = A\left(\frac{\sigma A}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} \tag{3.1b}$$

$$y_{pc}^* = \left(\frac{Y}{N}\right)^* = \left(\frac{Y}{L}\right)^* \vartheta(n) = y^* \frac{L}{N} = A\left(\frac{\sigma A}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} \vartheta(n) \quad \textit{income per-capita} \tag{3.1c}$$

$$c_{pc}^* = (1-\sigma) y_{pc}^* = (1-\sigma) A\left(\frac{\sigma A}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} \vartheta(n) \qquad \textit{cons per-capita} \tag{3.1d}$$

where δ is the (constant) depreciation rate of capital, σ the saving rate, and $\vartheta(n)=L/N$ is the fraction of the population in the work age span denoted by *(A,B)*. The work fraction (we will also term it the "active" fraction) $\vartheta(n)$ is defined as:

$$\vartheta(n)=\int_A^B c_n(a)da=\frac{\int_A^B e^{-na}p(a)da}{\int_0^L e^{-na}p(a)da} \qquad (3.2)$$

where a denotes individuals' age, and $c_n(a)$ is the stable age distribution of the population (the aforementioned *SAD*). The expression of the *SAD* shows that the form of the stable age distribution depends not only on the population growth rate in the stable regime, - the so called Lotka's or intrinsic growth rate in the demographic jargon (Keyfitz 1990), - but also on the *p(x) function,* denoting the survival function of the population.
The so called capital dilution effect (CD since now on) stands in the decreasing relation between income (and capital) per worker and the population growth rate, when the trivial identification between per-worker and per-capita quantities is made. The relations (3.1c)-(3.1d) show that this relation might be more complex when we avoid such identification by taking into account the presence of $\vartheta(n)$.

The active fraction ϑ as a function of n
Even if we disregard the effects due to the survival function, by keeping it fixed (a fact that certainly is unrealistic if we deal with epochs of demographic transitions), the evaluation of the impact of a given growth regime on welfare need to take into account the shape of the active fraction ϑ as a function of n. Fig. 1 reports the shape of $\vartheta(n)$ (and other population fractions) under a modern "western" survival function (the Italian female life table (LT) for 1999-2000, source ISTAT). The work age span is (15-65).

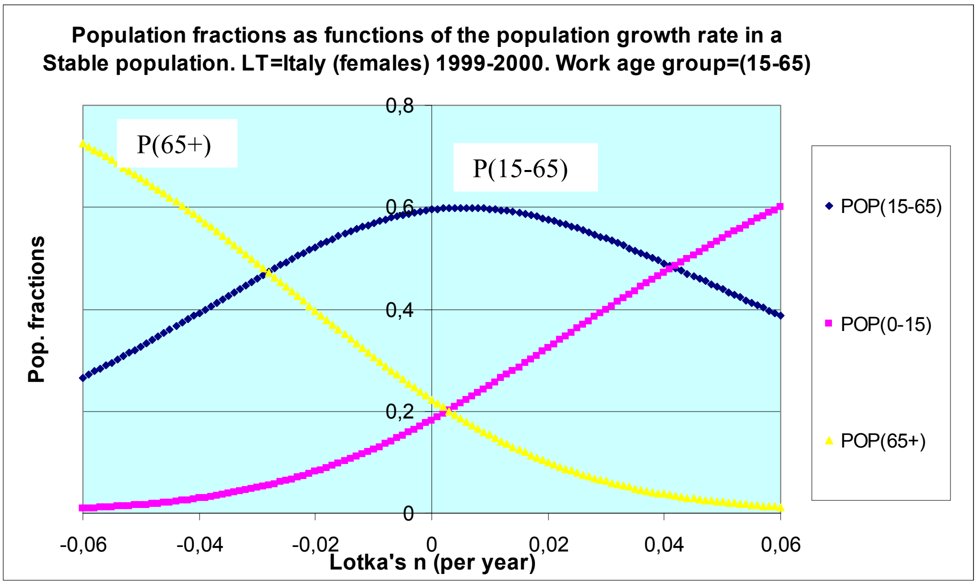

Fig. 1. Typical population fractions (adults=work, young, and retired) as functions of the population growth rate (LT is the Life Table from which the adopted survival curve is drawn)

The shape of $\vartheta(n)$ is influenced by both the ages *(A,B)* of entry/exits in the work age span and the shape of the survival function (for instance if the age of entry is very low then the maximal work fraction occurs in rather young populations, e.g. for fast growth, and vice-versa). Nevertheless the shape of $\vartheta(n)$ is always humped regardless of the peculiar demographic context (e.g. a given survival function) but fully general as the following result states:

Proposition. The work fraction is a humped function of n whatever be the shape of the survival function.

What are the consequences of taking into consideration the fraction of active population $\vartheta(n)$ for Solow's model ? It holds (Arthur and Mc Nicoll 1978, Blanchet 1988,1992)

$$\frac{dy_{pc}^{*}(n)}{dn} = y_{pc}^{*}(n)\left(-\frac{\alpha}{1-\alpha}(\delta+n)^{-1} + \left(A_N(n) - A_L(n)\right)\right) \qquad (3.3)$$

where the quantities $A_N(n)$and $A_L(n)$respectively denote the average age of the overall population and the average age of the population in the work age span (which in turn exhibit a significant variation with the population growth rate). The quantity:

$$CD(n) = -\frac{\alpha}{1-\alpha}(\delta+n)^{-1} < 0 \qquad (3.4)$$

has been called the "capital dilution" (CD) effect by Arthur and Mc Nicoll (1978), whereas the difference

$$IT(n) = A_N - A_L \qquad (3.5)$$

has been called the "intergenerational transfer" (IT) effect. The simpler manner to see that (3.5) defines an IT effect is to assume constant consumption rates over age. In this manner the fraction of active population is an estimate of the ratio between the consumption needs of the active (output-producing) population, and the consumptions needs of the overall population: thus a derivative of negative sign in the IT term implies that an increase in population growth value requires more resources being devoted to the "dependent population". In words: equation (3.3) shows that by properly taking population into account a new term, e.g. the IT effect, arises in addition to the traditional CD effect. Fig. 2 reports an illustration of CD and IT effects. The IT effect is a monotonically decreasing function of the population growth rate.

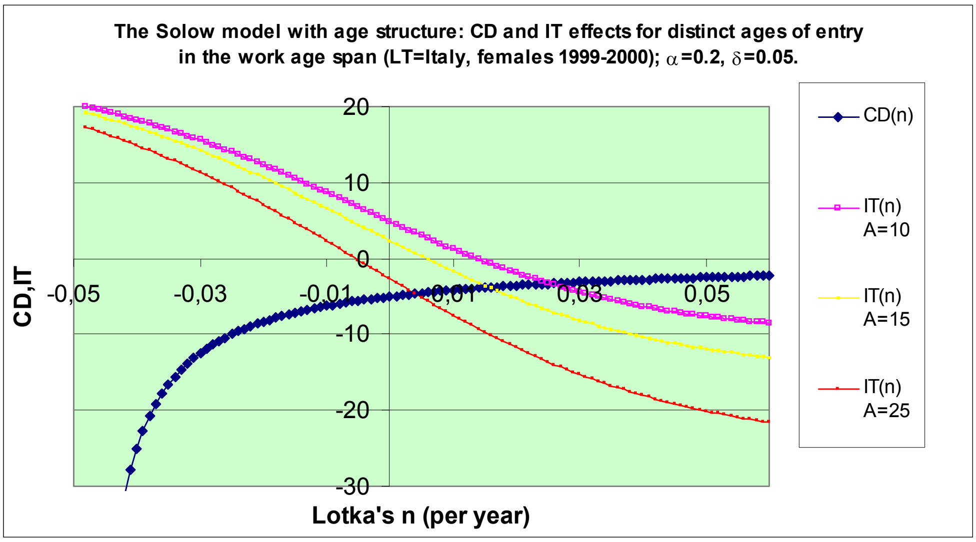

Fig. 2. The Solow model with age structure: CD and IT effects for distinct ages of entry in the work age span; LT=Italy 1999-2000, females; economic parameters: α=0.2, δ=0.05.

Onset of OPGR and WPGR

The balance between the CD and IT effects depends on several circumstances. Arthur and Mc Nicoll (1978) claimed that in most realistic cases it will be negative, e.g. the IT cumulate with Solow's CD effect, so that in general income per capita will be a declining function of population growth, thereby confirming Solow's proposition of capital dilution. Blanchet (1988,1992) conversely shows that it does not necessarily be so, e.g. Optimal Population Growth Rates are well possible even in simple cases. This is illustrated in fig. 3 reporting an illustration of the overall effect CD+IT. Fig. 3 shows that, notably, if an OPGR exists, then also a WPGR (Worst Population Growth Rate) necessarily exists (recall that the sum of the two effects gives the sign of the derivative of per capita income with respect to n). This fact, not pointed out by Blanchet (1988,1992), gives a feeling on the richness of the results provided by Solow's model, even with respect to the Samuelson-Deardoff debate. Notice that the WPGR is always located to the left of the OPGR (and it occurs for largely negative values of n). These results can be in some way taken in consideration for purposes of population policy (first best: move toward the OPGR value; second best: escape from the WPGR value).

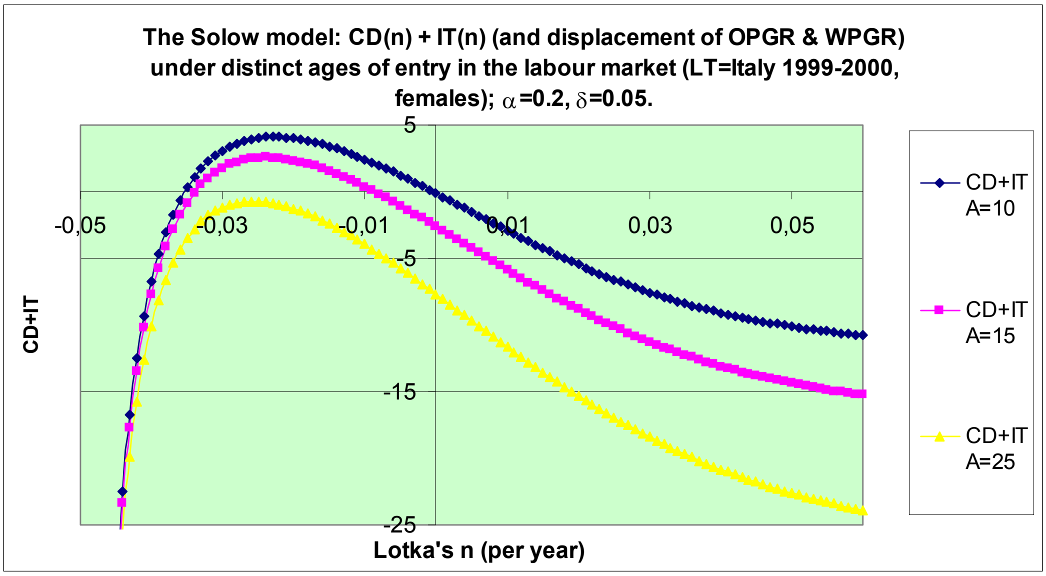

Fig. 3. The Solow model with age structure: balance of CD and IT effects for distinct ages of entry in the work age span, and onset of OPGR and WPGR (the adopted LT: Italy 1999-2000, females; economic parameters: $\alpha=0.2$*,* $\delta=0.05$*).*

As regards the major factors affecting the balance between the capital dilution and the intergeneration transfer effect, and in particular preventing/allowing the onset of OPGR and WPGR, it is easy to see that, given a prescribed survival curve, these are:

1. *Capital share*. Very high levels of the capital share (=very large α) can prevent the onset of an OPGR. (Indeed as α increases the ratio $\alpha/(1-\alpha)$ grows unbounded and thus also the CD effect does in absolute value. Since the IT effect is bounded, the CD effect dominates for very large α).
2. *Rate of capital depreciation*. High levels of δ reduce the CD effect and favour, coeteris paribus, the onset of an OPGR.
3. *Age of entrance in the labour market (A)*. For any n the work fraction $\vartheta(n)$ is a decreasing function of A. Thus any increase in A decreases the IT effect.

The effects of changes in the capital share are illustrated in fig. 4.

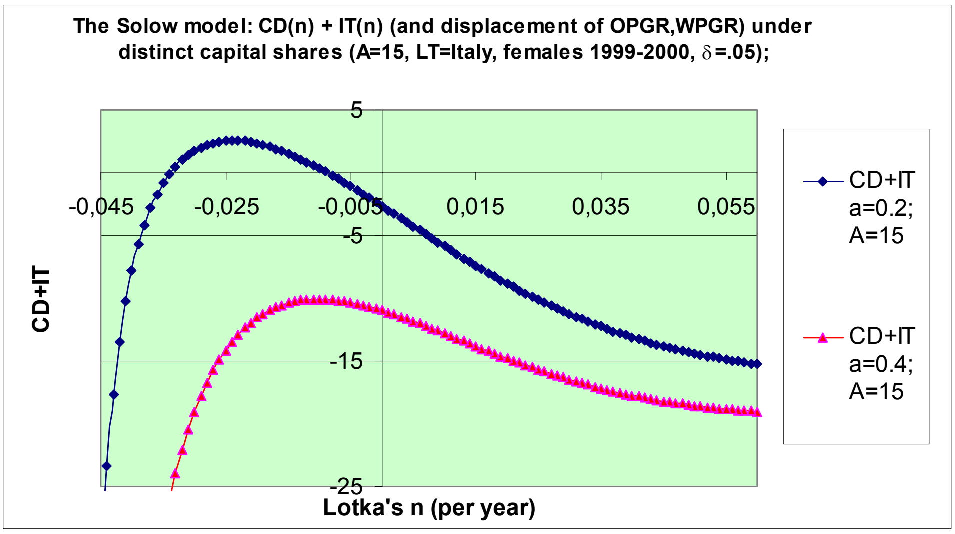

Fig. 4. The Solow model with age structure: balance of CD and IT effects for distinct levels of the capital share, and onset of OPGR and WPGR (A=15; LT: Italy 1999-2000, females; δ=0.05).

4. A Solow-type model with heterogeneity in the productivity profile by age

In this section we develop our basic framework, given by a Solow-Stiglitz-type model with stable population dynamics and heterogeneity in the productivity profile by age. For ease of comparison with Stiglitz (1969) multigroup model, we use a simplified version of our model using "discrete" age groups.

The age- productivity profile

Understanding age-productivity profiles is an issue of primary importance in economic growth research. For instance, if older individuals are less productive, an aging working population might yield significantly different performances compared to an economy in which productivity continuously increases with age, for instance via "learning by doing" effects.

The empirical evidence is controversial, although an "humped" productivity-age profile is the more common result. For example Skirbeck (2003, table 1) shows that, for 5 out of the 7 employer-employee studies[1], an inverted U-shaped work performance profile is found, where individuals in their 30s and 40s have the highest productivity levels. Employees above the age of 50 are found to have lower productivity than younger individuals, in spite of their higher wage levels[2]. However there also exist exceptions to the notion of decreasing productivity: Hellerstein and Neumark et al. (1995) suggest that productivity increases over the life span in a study of Israeli manufacturing firms.

[1] There are some main ways of measuring productivity by age: I) supervisors' ratings, piece-rate samples, 2) employer-employee matched data sets and 3)age-earnings data, as employment structure. In the second approach individual productivity is measured as the workers' marginal impact on the firm's value-added. This approach is likely to be less subjective than that based on supervisors' ratings, and there are fewer sample selection problems than studies on work-samples. However, the main challenge to this approach is to isolate the effect of the employees' age from all the other factors that affect the firm's value-added.

[2] We neglect here the existence of a wage-productivity discrepancy, largely documented in the literature (OECD 1998); indeed is known that the wage increases that last almost throughout a workers life seem to be determined by other factors than the workers' current productivity. Although productivity may fall in the latter half of the working life, wages continue to rise. This creates a discrepancy between productivity and wages, where younger workers are underpaid and older workers are overpaid relative to their productivity. The latter fact would introduce in the present model another channel of influence of the age structure on the growth.

The model
Compared to the Solow's model with stable population dynamics described in section 3, which is based on a standard Cobb-Douglas production function, we consider the following production function with heterogeneous labour

$$Y(t) = f(K(t), \Lambda(t)) \qquad (4.1)$$

where

$$\Lambda(t) = \sum_{i=1}^{m} h_i L_i(t) \qquad (4.2)$$

where h_i denotes the productivity factor of labour in age group i (say between ages (a_{i-1}, a_i)), m the number of age groups, and the number of workers in age group i is defined by:

$$L_i(t) = \int_{a_{i-1}}^{a_i} L(a,t)\,da \qquad (4.3)$$

where $L(a,t)$ is the (non-normalised) density of labour at exact age a. For simplicity we assume constant rates of participation and employment across ages, so that the labour weights

$$f_i(t) = \frac{L_i(t)}{L(t)} = \frac{\int_{a_{i-1}}^{a_i} L(a,t)\,da}{\int_{A}^{B} L(a,t)\,da} = \frac{\int_{a_{i-1}}^{a_i} L(a,t)\,da}{L(t)} \qquad (4.4)$$

are necessarily constant over time ($f_i(t) = f_i$) as they purely mirror the age profile of the population, which is constant by assumption. Since they depend on the state of growth of the population we write

$$f_i = f_i(n) \qquad (4.5)$$

Here we only focus on the following Cobb-Douglas-type form:

$$Y = AK^{\alpha}\Lambda^{1-\alpha} \qquad (4.6)$$

yielding:

$$y = \frac{Y}{L} = Ak^{\alpha}\left(\sum_{1}^{m} h_i f_i(n)\right)^{1-\alpha} = \left(\bar{h}(n)\right)^{1-\alpha} Ak^{\alpha} \qquad (4.7)$$

The relation (4.7) shows that, compared to the standard case of homogenous labour, output per worker is scaled by the average productivity factor $\bar{h}(n)$ which represents the average of the productivity profile over the workers' population. Thus it is a trivial matter to show that the model obeys an aggregate Solow-type equation. Since the presence of heterogeneous productivity has however also implications for distribution, we derive more carefully the basic equations, following Stiglitz (1969) Let us start from the basic balance equation of per capita-capital: $\dot{k} = saving\ per\ capita - (\delta + n)k = S/L - (\delta + n)k$. Let k_i denote the per-capita endowment of capital in the individual of group i, and $K_i= k_i\, L_i$ the total wealth in group i. For a constant saving rate σ one has

$$\frac{S}{L}=\frac{\sigma Y}{L} \qquad (4.8)$$

where

$$Y=\sum_{i=1}^{m}\left(w_i L_i + r k_i L_i\right)=\frac{\sigma}{L}\left(\sum_{i=1}^{m} w_i L_i + rK\right) \qquad (4.9)$$

Simple computations show that

$$w_i=(1-\alpha)y\frac{h_i}{\bar{h}(n)} \qquad (4.10)$$

and

$$r=\frac{\partial Y}{\partial K}=\alpha\frac{y}{k} \qquad (4.11)$$

One thus has

$$\frac{S}{L}=\frac{\sigma}{L}\left(\sum_{i=1}^{m} w_i L_i + rK\right)=\frac{\sigma}{L}\left(\sum_{i=1}^{m}(1-\alpha)y\frac{h_i}{\bar{h}(n)}L_i + rK\right)=\sigma y \qquad (4.12)$$

Thus the aggregate equation of per-capita capital becomes

$$\dot{k}=\sigma\left(\bar{h}(n)\right)^{1-\alpha}Ak^{\alpha}-(\delta+n)k \qquad (4.13)$$

which is a standard Solow's equation, and thus admits a unique state of balanced growth which is GAS. Its homogeneous counterpart is:

$$\dot{k}=\sigma h^{1-\alpha}Ak^{\alpha}-(\delta+n)k \qquad (4.14)$$

based on the production function with homogeneous labour $Y=AK^{\alpha}(hL)^{1-\alpha}$ where h is the constant productivity factor. Thus all the differences between (4.13) and (4.14) are incorporated in the function $\bar{h}(n)$. The following table reports the main equilibrium quantities of the extended Solow's model (4.13) (the known features of its counterpart (4.14) that are obtained by simply taking the parameter h as a constant independent of population growth and distribution, are those already listed in (3.1)):

$$k^*=\bar{h}(n)\left(\frac{\sigma A}{\delta+n}\right)^{\frac{1}{1-\alpha}} \qquad \delta+n>0 \qquad (4.15a)$$

$$y^*=\left(\bar{h}(n)\right)^{1-\alpha}A\left(k^*\right)^{\alpha}=A\bar{h}(n)\left(\frac{\sigma A}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} \qquad (4.15b)$$

$$y_{pc}^*=\left(\frac{Y}{P}\right)^*=\left(\frac{Y}{L}\right)^*\vartheta(n)=y^*\frac{L}{P}=A\bar{h}(n)\left(\frac{\sigma A}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}\vartheta(n) \qquad (4.15c)$$

$$c_{pc}^*=(1-\sigma)\left(\frac{Y}{P}\right)^*=(1-\sigma)A\bar{h}(n)\left(\frac{\sigma A}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}\vartheta(n) \qquad (4.15d)$$

There are three main question as regards (4.15):

a) which is the shape of $\bar{h}(n)$ as a function of the population growth rate, corresponding to any prescribed age profile of productivity ?
b) how does this profile affect Solow's equilibrium (4.15) ?
c) how different age profiles of productivity affect the equilibrium (4.15), for any given *n*, compared to the original homogeneous Solow's model (the "heterogeneity" issue) ?

Patterns of the $\bar{h}(n)$ *function*

The $\bar{h}(n)$ function has the following general definition (we get back to the continuous notation that allows an easier manipulation)

$$\bar{h}(n) = \int_A^B h(a) \frac{c_n(a)}{\int_A^B c_n(a) da} da = \int_A^B h(a) \frac{e^{-na} p(a)}{\int_A^B e^{-na} p(a) da} da \qquad (4.16)$$

In order to give specific answers to question a) above it is useful to introduce some special, but important, forms of the relation between productivity and age:

1. Linear learning by doing with age (LLbD)

Let $h(a) = \nu_0 + \nu_1 a \quad (v_1 > 0)$. In this case

$$\bar{h}(n) = \nu_0 + v_1 A_L(n) \qquad (4.17)$$

Since A_L is a monotonically decreasing function of *n*, then also $\bar{h}(n)$ is monotonically decreasing in *n*.

2. General learning by doing with age (GLbD)

Let *h(a)* be a general monotonically increasing function. The following general result holds (proof in appendix 1)

$$\frac{d\bar{h}(n)}{dn} = (-1)E(h)\left(\frac{E(a \bullet h) - E(a)E(h)}{E(h)}\right) = (-1)Cov_n(a,h) \quad (4.18)$$

Thus, if *h(a)* is a monotonically increasing (decreasing) function, then $\bar{h}(n)$ is a monotonically decreasing (increasing) function of *n* by the property of the Covariance as a statistical measure of concordance.

3. More general cases: ageing compensated learning by doing (ACLbD)

More general functions $h(a)$ are "humped shaped", as for instance in Miles (1999), or Skirbeck (2002), expressing both the LbD effect up to some given age, and the declining effect of ageing above that age. In this case it is possible to show that in most cases the $\bar{h}(n)$ factor inherits the humped form, e.g. it is a one humped function over the set (-∞,+∞) (obviously depending on actual economic circumstances the hump may occur for non "demographically plausible" n values). Fig. 5 below reports the shape of the $\bar{h}(n)$ factor when $h(a)$ has the form in Miles (1999).[3]

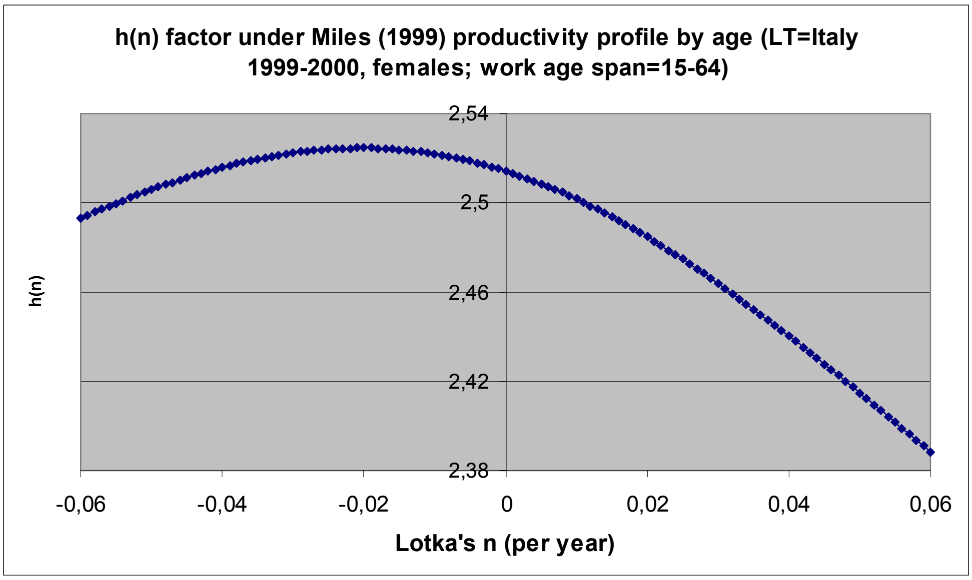

Fig. 5. The humped form of the $\bar{h}(n)$ productivity factor as a function of the population growth rate under Miles' (1999) productivity profile by age (LT=Italy 1999-2000, females; A=15, B=65).

The effect of age heterogeneity in productivity on Solow's equilibrium

As regards income per capita, the introduction of the age heterogeneity in productivity yields, additionally to the Capital Dilution (CD) and intergenerational transfer effect (IT) effects typical of the basic Solow's model, a productivity effect (PE). This may be seen by computing the log derivative of per-capita income (4.15), straightforwardly yielding

$$\frac{d\log\left(y^*_{pc}(n)\right)}{dn} = y^*_{pc}(n)\left(CD(n) + ITE(n) + PE(n)\right) \qquad (4.19)$$

where

$$PE(n) = \bar{h}'(n) = -Cov_n(a,h) \qquad (4.20)$$

Thus in some simple subcases it is possible to unambiguously sign the productivity effect:

Res. 1. Under GLbD with age the PE is always negative. Therefore the presence of GLbD always enhances Solow's result on capital dilution.

Though under learning by doing by age the productivity effect is always a declining function of the growth rate, it does not necessarily need to be so, as we have seen, under a general age profile of productivity. We do not discuss extensively the productivity effect in this paper since our major tasks is on the distribution problem tackled in the next section. We notice however that the presence of a productivity effect might in some cases quantitatively affect the location of OPGR and WPGR in a significant manner, compared to the "homogeneous" Solow's model.

[3] The equation for productivity estimated in Miles (1999, p. 13) has the following form: *h(a)=exp(0.05*a-0.0006*a²)*.

5. Income and wealth distribution

As first noted by Stiglitz (1969) the presence of heterogeneity in labour productivity is an "inequality preserving force". Here, compared to Stiglitz (1969) income and wealth groups are not based on some "ad hoc" static features but on age, which is a dynamic feature. We maintain all Stiglitz assumptions. It is possible to show that all Stiglitz relations hold, in particular saving per man in group i is:

$$s_i = b + \sigma\left(w_i + rk_i\right) \qquad (5.1)$$

where $b<0$ represents saving at zero income. The dynamics of wealth per capita in age group i is given by

$$\dot{k}_i = b + \sigma w_i + \left(r\sigma - (\delta + n)\right)k_i \qquad (5.2)$$

By (4.10) and (4.11)

$$\dot{k}_i = b + \sigma(1-\alpha)y\frac{h_i}{\bar{h}} + \left(\sigma\alpha\frac{y}{k} - (\delta + n)\right)k_i \quad i = 1,\dots,m \qquad (5.3)$$

As aggregate wealth per capita is $k = \sum_1^m k_i f_i$, simple manipulations lead to:

$$\dot{k} = b + \sigma y - (\delta + n)k \qquad (5.4)$$

which is the basic Stiglitz (1969) equation, and collapses for $b=0$ in the main Solow equation (4.13). We only consider here the case $b=0$. In this case the unique state of balanced growth k^* is GAS and the equations for the single age groups obey the following decoupled asymptotic equations

$$\dot{k}_i = \sigma(1-\alpha)y^*\frac{h_i}{\bar{h}} + \left(\sigma\alpha\frac{y^*}{k^*} - (\delta + n)\right)k_i \quad i = 1,\dots,m \qquad (5.5)$$

Notice that if $\sigma\alpha\frac{y^*}{k^*} - (\delta + n) < 0$, then all the m equations (5.5) have a meaningful equilibrium. As from Solow's balanced growth equilibrium it holds

$$\sigma\alpha\frac{y^*}{k^*} - (\delta + n) = \frac{1}{k^*}\left(\sigma\alpha y^* - (\delta + n)k^*\right) < \frac{1}{k^*}\left(\sigma y^* - (\delta + n)k^*\right) = 0 \qquad (5.6)$$

the aforementioned condition is always true. Thus the equilibrium in wealth per man in group i is:

$$k_i^* = \frac{\sigma(1-\alpha)y^*}{\delta + n - \sigma\alpha\frac{y^*}{k^*}} \cdot \frac{h_i}{\bar{h}} \quad i = 1,\dots,m \qquad (5.7)$$

Expressions (5.7) provide the equilibrium distribution of wealth (C) in the various age groups of workers:

$$C = \begin{cases} k_1^* & \cdots & \cdots & k_m^* \\ f_1(n) & \ldots & \cdots & f_m(n) \end{cases} \qquad (5.8)$$

The corresponding equilibrium distribution of income in workers age groups is easily found by the relationships

$$y_i^* = w_i + rk_i^* \quad i = 1,\ldots,m \qquad (5.9)$$

For sake of brevity we do not report the corresponding distributions of consumptions and saving given that, since the saving rate is constant across age in this extremely simplified model, the age profile of consumption and saving mirror that of income (they would not necessarily do so under an age related saving profile). Notice that the previous expressions little say about patterns of income and wealth distribution in the overall population.

Remark. By simple algebra on the previous relations one can express wealth (and also income) in age group *i* as a linear function of productivity, obtaining

$$k_i^* = \rho(n) \cdot h_i \quad i = 1,\ldots,m \qquad (5.10)$$

where

$$\rho(n) = \frac{\sigma(1-\alpha)y^*(n)}{\delta + n - \sigma\alpha \dfrac{y^*(n)}{k^*(n)}} \cdot \frac{1}{\overline{h}(n)} \quad i = 1,\ldots,m \qquad (5.11)$$

Thanks to linearity one can determine the distribution of the k_i^* for any given distribution of the h_i as done in Stiglitz (1969). The interest of the present case lies in the fact that the h_i's are functions of age and thus in the possibility of relating the age distribution of wealth at equilibrium with the age distribution of the population.

The wealth distribution among workers in the equilibrium of balanced growth

As we are basically interested in the relation between regimes of population growth and age distribution (as summarised by *n*) on the one hand, and wealth and income distribution on the other hand, we must look for the true individuals' distributions of income and wealth in the overall workers' population. The task of studying the degree of inequality in individuals' income and wealth can be carried out by resorting, after having determined the individuals' income (wealth) distribution from the age distribution of income (wealth), to either synthetic measures of inequality such as the Gini's index, or to general "distributional" measures of inequality, such as the Lorenz curve. For sake of simplicity we limit ourselves here to Gini's index. Fig. 6 reports for every value of the population growth rate, the value of Gini's index associated to the distribution of income among workers corresponding to the given growth regime, under the following parameter constellation: α*=0.2,* δ*=0.2, Q=1, s=0.2, A=20, B=65*, *LT=Italy 199-2000, females*.

The inspection of fig. 6 reveals that: i) a Minimal Inequality Population Growth Rate (MinIPGR) does exist (with a value around 2%); ii) two Maximal Inequality Population Growth Rates (MaxIPGR) also exist. In the current simulation, just drawn for illustrative purposes, both MaxIPGR occur for implausibly large values of the population growth rate, but this does not exclude that plausible values occur under alternative parameter constellations.

To summarise: the current illustration has shown that additionally to the "efficient" PGR (e.g. the OPGR described in section 3), also an "equity optimal" PGR exists. This raises the question of whether this "equity optimal" population growth rate occurs in the same region of values as the

"efficient" PGR (e.g. the OPGR described in section 3), or they are significantly different (e.g. is there a trade off between equity and efficiency as regards the corresponding optimal PGRs ?). To answer the question we go back to the corresponding relation between per capita income and the population growth rate (drawn under the same parameter constellation), reported in fig. 7. Fig. 7 shows that the efficient PGR is close to zero, e.g. 2% less compared to the "equity optimal" PGR. But two percentage points in terms of population growth rate represent a huge difference ! This makes it clear that for economies as those described here the achievement of both efficiency and equity is an unfeasible task.
As far as we know the latter results have never been pointed out in the literature on growth and distribution.

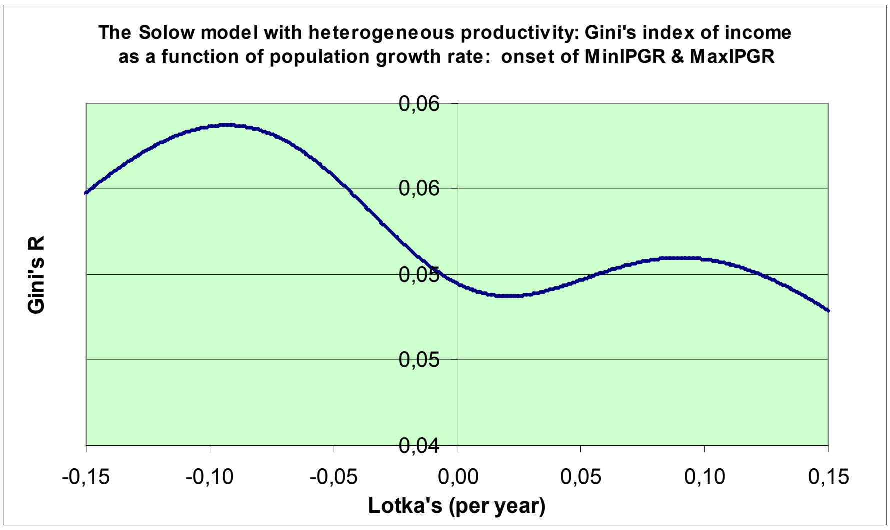

Fig. 6. Gini's index of income distribution as a function of the population growth rate. Onset of MinIPGR and MaxIPGR. Parameters: α=0.2, δ=0.2, Q=1, s=0.2, A=20, B=65, LT=Italy 199-2000, females.

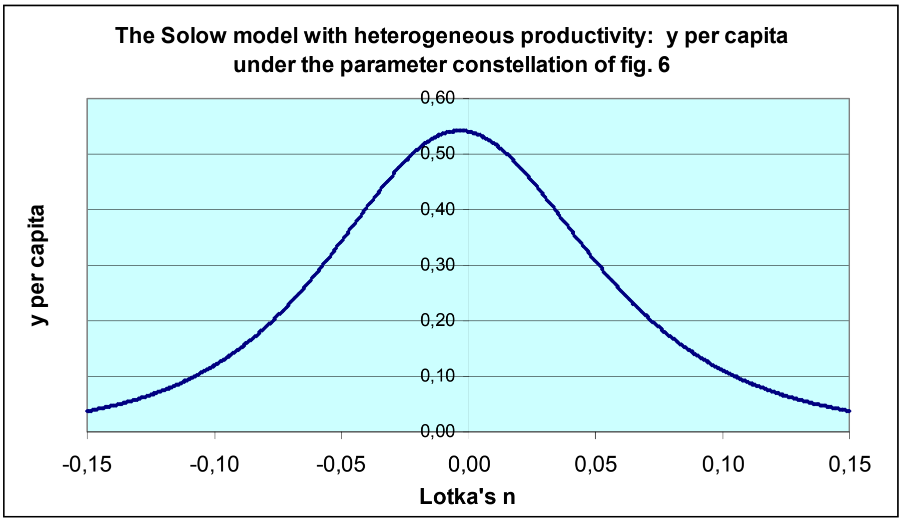

Fig. 7. Income per capita as a function of of the population growth rate under the same parameter constellation adopted in fig. 6.

6. Conclusive remarks.

The present paper develops a descriptive Solovian growth model including the age structure of the population and the dynamics of wealth among age groups according to Stiglitz (1969). Such an age structure is responsible for the so-called dilution and intergenerational effects, which have been discussed in the literature on the existence of an optimal population growth rate. However the literature has been not systematically concerned with the theoretical implications of age structure with respect to other economic factors different from those effects. In this paper we extended the previous model, suggesting that age structure may affect economic growth and welfare through other avenues, first via the age productivity profiles.

We study existence and stability of the balanced growth path. The model allows to neatly clarify the effects played by age structure and the growth rate of the population: 1) on economic growth and 2) on the individual distributions of wealth and income.

It appears that age structure significantly affects 1) per-capita capital and per-capita income (although in an unexpectedly different manner); 2) the individual income distribution. The main message is twofold: 1) an optimal population growth – and also a worst population growth - rate exists both for the economic growth and welfare; 2) a Minimal Inequality PGR exists. As to first point, this means that many policy implications widely recognised and implemented in order to reduce poverty in developing countries, namely to reduce fertility as far and faster as possible, could be wrong. Conversely a framework as the present allows for a first attempt evaluation, throuh realistic calibration of the model, to evaluate whether there is any real need to spend a lot of resources to exit from LLF.

As to the second point, the main consequence is that critical values of the PGR are different depending on whether they regard the issue of the efficient PGR (the so called OPGR) or the more "equitable" PGR (the so-called MIPGR), bringing about a possible complicated trade-off between efficiency and equity emerging from the demographic policy.

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Appendix . Proof of result (4.18)

Let:

$$\overline{h}(n)=\int_A^B h(a)\frac{c_n(a)}{\int_A^B c_n(a)da}da$$

We have

$$\frac{d\overline{h}(n)}{dn}=\overline{h}(n)\left(A_L-\frac{\int_A^B ah(a)c_n(a)da}{\int_A^B h(a)c_n(a)da}\right)=\overline{h}(n)\left(A_L-\frac{\int_A^B ah(a)c_n(a)da}{\int_A^B c_n(a)da}\frac{\int_A^B c_n(a)da}{\int_A^B h(a)c_n(a)da}\right)=$$

$$=\overline{h}(n)\left(A_L-\frac{E(a\bullet h)}{E(h)}\right)$$

But the quantities A_L and $\overline{h}(n)$ are expectations (or mean values) over the age distribution of the workers' population, namely the mean age and the mean productivity of the workers' population:

$$A_L=E(a)\ ;\ \ \overline{h}(n)=E(h)$$

Thus

$$\frac{d\overline{h}(n)}{dn}=(-1)E(h)\left(\frac{E(a\bullet h)-E(a)E(h)}{E(h)}\right)=(-1)Cov_n(a,h)$$

Thus if *h(a)* is a monotonic transform of age we conclude that

$$\frac{d\overline{h}(n)}{dn}<0$$