Macroeconomic Volatility and Income Inequality in a Stochastically Growing Economy

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Abstract

This paper employs a stochastic growth model to analyze the effect of macroeconomic volatility on the relationship between income distribution and growth. We initially characterize the equilibrium and show how the distribution of income depends upon two factors: the initial distribution of capital, and the equilibrium labor supply and find that an increase in volatility raises the mean growth rate and income inequality. We calibrate the model using standard parameter estimates, and find that it successfully replicates key growth and distributional characteristics. The latter part of the paper uses this framework to analyze the design of tax policy to achieve desired growth, distribution, and welfare objectives. Two general conclusions clearly emerge. First, increasing (average) welfare and the growth rate does not necessarily entail an increase in inequality. Second, fiscal policy often has conflicting effects on the distributions of gross and net income, with changes in factor prices resulting in greater pre-tax inequality but the redistributive effect of taxes yielding a more equal post-tax distribution.

November 2003

This paper has benefited from comments received from seminar presentations at Nuffield College, Oxford, and the University of Munich, the 2003 EEA meetings, and the 2003 IAES meetings.
1. Introduction

The relationship between income inequality and economic growth is both important and controversial. Knowledge of how these two variables are related is essential if a policymaker is to be able to neutralize any undesired consequences of, say, a growth-enhancing policy on the economy’s distribution of income. Its controversy derives from the fact that it has been difficult to reconcile the different theories determining the relationship, especially since the empirical evidence has not been able to provide clear support for one or another approach.\footnote{See, among others, Stiglitz (1969), Bourguignon (1981), Aghion and Bolton (1997) andGalor and Tsiddon (1997), for theoretical work; Alesina and Rodrik (1994), Persson and Tabellini (1994), and Forbes (2000), for empirical analyses; and Aghion, Caroli, and García-Peñalosa (1999) for an overview.} Despite the controversy, one thing is clear. Both the degree of income inequality and the growth rate are endogenous variables and thus need to be jointly determined as equilibrium outcomes within a consistently formulated macroeconomic growth model.

Empirical evidence suggests that macroeconomic volatility is potentially an important channel through which income inequality and growth may be mutually related. For example, it is well-known that the middle-income Latin American economies are associated with much greater income inequality than are the East-Asian “tigers”. In 1990, the Gini coefficients of the distribution of income in Brazil, Chile, Mexico, and Venezuela ranged between 55-64%, while those of Hong Kong, Korea, Taiwan, and Singapore, were between 30 and 41%. At the same time, the former were subject to much greater fluctuations in their respective growth rates than were the latter: during the 1980s, the standard deviation of the rate of output growth was, on average, 5.9% for the four Latin American economies, and 2.8% for the East Asian countries.\footnote{See Breen and García-Peñalosa (2002) for a description of the data sources used for these calculations.} In fact, using a broader set of data, Breen and García-Peñalosa (2002) obtain a positive relationship between a country’s volatility (measured by the standard deviation of the rate of GDP growth) and income inequality. On the other hand, simple growth regressions have suggested that the mean growth rate is related to its volatility, although the results are not uniform with respect to the direction of this relationship.\footnote{For example, using a sample of 92 countries, Ramey and Ramey (1995) obtained a negative relationship between inequality and growth, in contrast to earlier work by Kormendi and Mequire (1985) who obtained a positive relationship.}

Our objective in this paper is to model a mechanism through which aggregate risk -- which
we shall attribute to production shocks -- has a direct impact on distribution. We employ an extension of the stochastic growth model developed by Grinols and Turnovsky (1993, 1998), Smith (1996), Corsetti (1997) and Turnovsky (2000b). This is a one-sector growth model in which, due to the presence of an externality stemming from the aggregate capital stock, equilibrium output evolves in accordance with a stochastic AK technology. Previous studies have been unable to analyze the impact of volatility on income distribution, as they either abstract from labor, or otherwise, assume that agents are identical in all respects. We introduce the assumption that agents are heterogeneous with respect to their initial endowment of capital. As a result, the labor supply responses to different degrees of risk will induce changes in factor prices and affect the distribution of income.

Adopting this framework, the equilibrium growth rate, its volatility, and the distribution of income become jointly determined. Our analysis proceeds in several stages. To start with, we derive the equilibrium balanced growth path in a stochastic growth model with given tax rates. We show how this equilibrium has a simple recursive structure. First, the mean growth rate and labor supply (employment) are determined at the point of intersection of two tradeoff locuses that characterize the equilibrium relationship between rates of return and product market equilibrium, respectively. The distribution of income is shown to depend upon two factors: the initial distribution of capital, and the equilibrium labor supply (and hence risk). We find that an increase in production risk raises the mean growth rate, its volatility, and the degree of income inequality. We next calibrate the model using standard parameter estimates, typical of a developed economy.

The latter part of the paper uses this framework to analyze the effect of first-best taxation. As it is well-known, the externality associated with the capital stock implies that the competitive growth rate is too low. The first-best can then be attained through suitable taxes and subsidies. When agents are heterogeneous, the use of growth enhancing policies raises the question of what is the impact of first-best policies on the distribution of income. Two general conclusions emerge from our analysis. First, increasing (average) welfare and the growth rate does not necessarily entail an increase in inequality. In particular, we find that fiscal policy has conflicting effects on the distributions of gross and net income. First-best policies result in changes in factor prices that increase pre-tax inequality, but the direct redistributive effect of taxes tends to yield a more equal post-tax distribution. Second,
the positive correlation between volatility and distribution may disappear when first-best policies are implemented. Greater risk, by raising the labor supply and hence the wage bill, requires a lower wage tax, thus making the tax system more progressive. This effect can be strong enough to offset the positive effect of risk on inequality.

Research on the macroeconomic determinants of income inequality has mainly focused on three aspects: growth, trade, and inflation. Studies of the impact of growth on distribution range from the studies of the impact of structural change, such as the Kuznets hypothesis, to theories of skill-biased technical change. Based on Heckscher-Ohlin theory, international trade has been argued to be a major determinant of distribution, and this aspect has recently acquired prominence in the debate on the increase in inequality in a number of industrialized countries. One of the most consistently supported empirical correlations is that between inflation and inequality, explained by the fact that because inflation is a regressive tax, it generates greater inequality.\(^4\) Our paper seeks to introduce a new and so far ignored factor, the degree of risk in an economy, into the analysis of inequality.

The paper is related to Alesina and Rodrik (1994), Persson and Tabellini (1994), and Bertola (1993), who develop (non-stochastic) AK growth models in which agents differ in their initial stocks of capital. The first two papers have, however, a very different focus as they take initial inequality as given and examine how it affects the rate of growth. Bertola (1993) is closer to our approach in that he emphasizes how technological parameters, specifically the productivity of capital, jointly determine distribution and growth. He also examines how policies directed at increasing the growth rate affect the distribution of consumption, although his assumption of a constant labor supply implies that the distribution of income is independent of policy choices. Our approach shares with these three papers an important limitation, namely, that the assumption that agents differ only in their initial stocks of capital implies that there are no income dynamics. A more general study of heterogeneity and the dynamics of distribution in growth models can be found in Caselli and Ventura (2000).

The paper closest to our work, at least in spirit, is Aghion, Banerjee, and Piketty (1999), who find that greater inequality is associated with more volatility. They show how combining capital

\(^4\) See Okun (1971) and Taylor (1981), and more recently, Albanesi (2002).
market imperfections with inequality in a two-sector model can generate endogenous fluctuations in output and investment. In their model it is unequal access to investment opportunities and the gap between the returns to investment in the modern and the traditional sectors that cause fluctuations. We reverse the focus, examining how exogenous production uncertainty determines output volatility and income distribution.

The paper is organized as follows. Section 2 presents the model and derives the equilibrium growth rate, labor supply, and volatility. Section 3 examines the determinants of the distribution of income. Section 4 shows, analytically and numerically, that in the absence of taxation greater risk is associated with a more unequal distribution of income. Section 5 starts by obtaining the first-best optimum, and shows that the competitive growth rate is too low. It is followed by an analysis of first-best taxation. Numerical analysis is then used to illustrate the distributional implications of the various policies. Section 6 concludes, while technical details are provided in the Appendix.

2. The Model

2.1 Description of the economy

Technology and factor payments

Firms shall be indexed by $j$. We assume that the representative firm produces output in accordance with the stochastic Cobb-Douglas production function

$$dY_j = A(L_j K)^\alpha K_j^{1-\alpha} (dt + du)$$

where $K_j$ denotes the individual firm’s capital stock, $L_j$ denotes the individual firm’s employment of labor, $K$ is the average stock of capital in the economy, so that $L/K$ measures the efficiency units of labor employed by the firm; see e.g. Corsetti (1997). The stochastic variable is temporally independent, with mean zero and variance $\sigma^2 dt$ over the instant $dt$. The stochastic production function exhibits constant returns to scale in the private factors -- labor and the private capital stock.

All firms face identical production conditions and are subject to the same realization of an economy-wide stochastic shock. Hence they will all choose the same level of employment and
capital stock. That is, \( K_j = K \) and \( L_j = L \) for all \( j \), where \( L \) is the average economy-wide level of employment. The average capital stock yields an externality such that in equilibrium the aggregate (average) production function is linear in the aggregate capital stock, as in Romer (1986), namely

\[
dY = AL^\alpha K (dt + du) \equiv \Omega(L)K(dt + du) \tag{1b}
\]

where \( \Omega(L) \equiv AL^\alpha \) and \( \partial \Omega / \partial L > 0 \).

We assume that the wage rate, \( z \), over the period \((t, t+dt)\) is determined at the start of the period and is set equal to the expected marginal physical product of labor over that period. The total rate of return to labor over the same interval is thus specified nonstochastically by

\[
dZ = zdt = \left( \frac{\partial F}{\partial L_j} \right)_{K_j = K, L_j = L} dt . \tag{2a}
\]

where \( z = \alpha \Omega L^{\alpha-1} K \equiv wK \).

The private rate of return to capital, \( dR \), over the interval \((t, t+dt)\) is thus determined residually by

\[
dR = \frac{dY - LdZ}{K} \equiv rdt + du_K \tag{2b}
\]

where \( r \equiv \left( \frac{\partial F}{\partial K_j} \right)_{K_j = K, L_j = L} = (1 - \alpha) \Omega \), and \( du_K \equiv \Omega du \).

These two equations assume that the wage rate, \( z \), is fixed over the time period \((t, t+dt)\), so that the return on capital absorbs all output fluctuations. The rationale for this assumption is that in industrial economies wages are usually fixed ex ante, while the return to capital is, at least in part, determined ex post and thus absorbs most of the fluctuations in profitability.\(^5\) Differentiating the production function and given that firms are identical, we find that the equilibrium return to capital is independent of the stock of capital while the wage rate is proportional to the average stock of capital, and therefore grows with the economy.\(^6\) In addition, we have \( \partial r / \partial L > 0 \) and \( \partial w / \partial L < 0 \).

\(^5\) In the United States, for example, the relative variability of stock returns over the period 1955-1995 were around 32% per annum, while the relative variability of wages over that same period was only 2%.

\(^6\) Intuitively, in a growing economy, with the labor supply fixed, the higher income earned by labor is reflected in higher returns, whereas with capital growing at the same rate as output, returns to capital remain constant.
reflecting the fact that more employment raises the productivity of capital but lowers that of labor.

Consumers

There is a mass 1 of infinitely-lived agents in the economy. Consumers are indexed by $i$ and are identical in all respects except for their initial stock of capital, $K_{i0}$. Since the economy grows, we will be interested in the share of individual $i$ in the total stock of capital, $k_i$, defined as $k_i = K_i/K$, where $K$ is the aggregate (or average) stock. Relative capital has a distribution function $G(k_i)$, mean $\sum_i k_i = 1$, and variance $\sigma_k^2$.

All agents are endowed with a unit of time that can be allocated either to leisure, $l_i$, or to work, $1 - l_i = L_i$. A typical consumer maximizes expected lifetime utility, assumed to be a function of both consumption and the amount of leisure time, in accordance with the isoelastic utility function

$$\max E_U \int_0^\infty \frac{1}{\gamma} \left( C_i(t) l_i^\eta \right)^\gamma e^{-rt} dt, \quad \text{with} \quad -\infty < \gamma < 1, \eta > 0, \gamma \eta < 1$$

where $1 - \gamma$ equals the coefficient of relative risk aversion. Empirical evidence suggests that this is relatively large, certainly well in excess of unity, so that we shall assume $\gamma < 0$. The parameter $\eta$ represents the elasticity of leisure in utility. This maximization is subject to the agent's capital accumulation constraint

$$dK_i = (1 - \tau) K_i dt + (1 - \tau') K_i du_k + (1 - \tau_w) w(1 - l_i) K dt$$

$$- (1 + \tau_c) C_i dt + sE(dK_i) + s'(dK_i - Ed(K_i))$$

with $du_k = \Omega du$. The fiscal instruments used by the government are a subsidy on investment in physical capital, at rates $s$ and $s'$ for the deterministic and the stochastic component of investment, respectively; a consumption tax, $\tau_c$; a wage tax, $\tau_w$; a tax on the deterministic component of capital income, $\tau_k$, and a tax on the stochastic component of capital income, $\tau_k'$. Taking expectations of this expression and substituting back for $E(dK_i)$, we can express this budget constraint as

$$dK_i = \frac{(1 - \tau) r K_i dt + (1 - \tau') w(1 - l_i) K - (1 + \tau_c) C_i}{1 - s} dt + K_i \frac{1 - \tau_k'}{1 - s'} du_k$$

Some of the empirical estimates supporting this assumption are noted in Section 4.2 below.
It is important to observe that with the equilibrium wage rate being tied to the aggregate stock of capital, the rate of accumulation of the individual’s capital stock depends on the aggregate stock of capital, which in turn evolves according to

$$\begin{align*}
dK &= \left(1 - \tau_k \right) r + (1 - \tau_w) w(1 - l) \frac{K}{1 - s} - (1 + \tau_c) C \frac{1 - \tau'_k}{1 - s'} \; du_k
\end{align*}$$

(4b)

where \( l \) denotes the average (aggregate) fraction of time devoted to leisure. The agent therefore needs to take this relationship into account in performing her optimization.

**Government policy**

The government balances the public budget each period, implying

$$\begin{align*}
sE(dK) dt + s' \frac{1 - \tau'_k}{1 - s'} K du_K &= \left[ \tau_c C + \tau_k rK + \tau_w w(1 - l) K \right] dt + \tau'_k K du_K
\end{align*}$$

(5)

Note that both expenditures and tax receipts have a deterministic and a stochastic component. Equating them respectively yields the following constraints required to maintain a balanced budget:

$$\begin{align*}
\tau'_k &= s', \quad (6a) \\
\tau_k r + \tau_w w(1 - l) + \tau_c \frac{C}{K} &= s\psi. \quad (6b)
\end{align*}$$

Two points should be noted. First, some of the taxes may be negative, in which case they become subsidies, in addition to the investment subsidy. Second, the assumption that the deterministic and stochastic components of income are taxed at different rates requires that the agent (and the tax authority) disentangle the deterministic from the stochastic components of income, something that may not be unlikely in practice.\(^8\) However, this assumption is mainly made for analytical simplicity. Taxing both components at the same rate, with the government using public debt in order to compensate any surplus or deficit, would not change any of our results, since as we will see below, the tax rates on the stochastic component of capital income does not affect any of the

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\(^8\) One rationale for differential taxes on the deterministic and stochastic components of capital income is to identify the latter with unanticipated capital gains, which typically, are taxed at a different rate.
equilibrium relationships.\(^9\)

### 2.2 Consumer optimization

The consumer’s formal optimization problem is to maximize (3) subject to equations (4a) and (4b). The first-order conditions with respect to consumption and leisure yield

\[
\frac{1}{C_i} \left( \frac{C_i}{l_i} \right)^\gamma = \frac{1+\tau_i}{1-s} X_{K_i},
\]

\[
\frac{\eta}{l} \left( \frac{C_i}{l_i} \right)^\gamma = \frac{1-\tau_i w}{1-s} w K X_{K_i}
\]

where \(X(K, K)\) is the value function and \(X_{K_i}\) its derivative with respect to \(K_i\) (see Appendix).

In the Appendix we show that utility maximization implies that the dynamic evolution of the stock of capital of agent \(i\) is given by

\[
\frac{dK_i}{K_i} = \left[ \frac{r(1-\tau_i)/(1-s) - \beta}{1-\gamma} - \frac{\gamma^2 \Omega^2 \sigma^2}{2} \right] dt + \Omega du \equiv \psi dt + \Omega du,
\]

where \(\Omega\) and \(r\) are defined in equations (1) and (2). There we have expressed them as functions of equilibrium employment, \(L\), but assuming that the aggregate labor market clears, yields

\[
\sum_j L_j = L = \sum_i (1-l_i)
\]

and we can equally well write \(\Omega\) and \(r\) as functions of \((1-l)\).\(^{10}\) From (8) we see that the rate of growth of capital -- and therefore of output -- has a deterministic and a stochastic component, so that the average growth rate, \(\psi\), and its standard deviation, \(\sigma_{\psi}\), are respectively

\[
\psi = \frac{r \left( \frac{1-\tau_i}{1-s} \right) - \beta}{1-\gamma} - \frac{\gamma^2 \Omega^2 \sigma^2}{2} \quad \text{and} \quad \sigma_{\psi} = \Omega \sigma
\]

Observe that the only difference between agents, namely their initial stock of capital, does not appear in this equation. Hence all individuals choose the same rate of growth of their stock of capital.

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\(^9\) See Turnovsky (2000b) and García Peñalosa and Turnovsky (2002).

\(^{10}\) Thus we may write \(\Omega(l) = A(1-l)^\alpha\) and \(r = (1-\alpha)\Omega(l)\), where \(\Omega'(l) < 0\).
capital, $\psi$. This has two implications. First, the aggregate rate of growth of capital is identical to the individual rate of growth and unaffected by the initial distribution of endowments, hence

$$dK/K = \psi dt + \Omega du.$$  \hfill (8')

Second, since the capital stock of all agents grows at the same rate, the distribution of capital endowments does not change over time. That is, at any point in time, the wealth share of agent $i$, $k_i$, is given by her initial share $k_i,0$, say.

Dividing equation (7a) by (7b), we obtain the consumption to capital ratio of agent $i$,

$$\frac{C_i}{K_i} = \frac{w(1-\tau_w)l_i}{\eta(1+\tau_c)k_i}.$$  \hfill (11)

Aggregating over the individuals and noting that $\sum_i k_i = 1$, $\sum_i l_i = l$, the aggregate economy-wide consumption-capital ratio is

$$\frac{C}{K} = \frac{w(1-\tau_w)l}{\eta(1+\tau_c)}.$$  \hfill (11')

In addition, the following transversality condition must hold

$$\lim_{t \to \infty} E\left[K_i(t) e^{-\beta t}\right] = 0.$$  \hfill (12)

With $K_i(t)$ evolving in accordance with the stochastic path (8), (12) can be shown to reduce to\(^\text{11}\)

$$\gamma \left[ r \frac{1-\tau_k}{1-s} - \frac{\gamma}{2} (1-\gamma) \Omega^2 \sigma^2 \right] < \beta$$

which, when combined with (10), can be shown to be equivalent to the condition

$$r \left( \frac{1-\tau_k}{1-s} \right) > \psi$$  \hfill (13)

i.e. the equilibrium rate of return on capital must exceed the equilibrium growth rate. Dividing the aggregate accumulation equation, (4b), by $K$, this condition can also be shown to be equivalent to

\(^{11}\) See Turnovsky (2000b).
(1 + τ_c) \frac{C}{K} > (1 - \tau_w)w(1 - l) \quad (13')

implying that part of income from capital is consumed.\(^\text{12}\) Combining with (9'), this can be further expressed as

\[ l > \frac{\eta}{1 + \eta} \quad (13'\)')

Recalling the individual budget constraint, (4a), we can write the individual’s mean rate of capital accumulation as

\[ \frac{E(dK_i / K_i)}{dt} = \frac{1 - \tau_K}{1 - s} w + \frac{1 - \tau_w}{1 - s} \frac{1 - l_i}{k_i} - \frac{1 + \tau_c}{1 - s} \frac{C_i}{K_i} \quad (14) \]

Together with equation (11), this expression implies that agent \( i \)'s supply of labor is

\[ 1 - l_i = \frac{1}{1 + \eta} \left[ 1 - \eta \left( \frac{1 - \tau_k}{1 - \tau_w} \frac{r(1 - \tau_k)/(1 - s) - \psi}{w} \right) k_i \right] \quad (15) \]

Noting the transversality condition, \( r(1 - \tau_k)/(1 - s) > \psi \), (15) implies that an increase in the agent’s capital (wealth) has a negative effect on her labor supply; wealthier individuals chose to “buy” more leisure. In effect, they compensate for their larger capital endowment, and the higher growth rate it would support, by providing less labor and having an exactly offsetting effect on the growth rate.

Because the rate of growth is the same for all agents, individual labor supplies are linear in the wealth shares of agents. The aggregate labor supply, \( 1 - l = 1 - \sum_i l_i \), is then independent of the initial distribution of capital. Summing equation (15) over the agents and using the fact that \( \sum_i k_i = 1 \), we obtain the aggregate labor supply relation,

\[ 1 - l = \frac{1}{1 + \eta} \left[ 1 - \eta \frac{1 - s}{1 - \tau_w} \frac{r(1 - \tau_k)/(1 - s) - \psi}{w} \right] \quad (15') \]

and combining (15) and (15') we can derive the following expression for the “relative labor supply”

\(^{12}\) This latter condition reduces to \( C/K > 0 \) in the original Merton (1969) model, which abstracted from labor income.
\[ l_i - l = \left( l - \frac{\eta}{1+\eta} \right)(k_i - 1) \] (15’’)

Again we see that the transversality condition, now expressed as (13’’), implies a positive relationship between relative wealth and leisure.

2.3. **Macroeconomic equilibrium**

The key equilibrium relationships can be summarized by

**Equilibrium growth rate**

\[ \psi = r \left( 1 - \frac{1 - \tau_k}{1 - s} \right) \beta - \frac{\gamma}{2} \Omega^2 \sigma^2 \] (16a)

**Equilibrium Volatility**

\[ \sigma_\psi = \Omega \sigma \] (16b)

**Individual consumption-capital ratio**

\[ \frac{C_i}{K_i} = \frac{w (1 - \tau_w) l_i}{\eta (1 + \tau_c) k_i} \] (16c)

**Aggregate consumption-capital ratio**

\[ \frac{C}{K} = \frac{w (1 - \tau_w) l}{\eta (1 + \tau_c)} \] (16d)

**Individual Budget Constraint**

\[ \psi = r \frac{1 - \tau_k}{1 - s} + w \frac{1 - \tau_w}{1 - s} \frac{l_i}{k_i} + \frac{1 + \tau_c}{1 - s} \frac{C_i}{K_i} \] (16e)

**Goods market equilibrium**

\[ \psi = \Omega - \frac{C}{K} \] (16f)
Government budget constraint

\[ \tau_k r + \tau_w w(1-l) + \tau_c \frac{C}{K} = s\psi \]  

Recalling the definitions of \( r(l), \) \( w(l), \) and \( \Omega(l), \) and given \( k_i, \) these equations jointly determine the individual and aggregate consumption-capital ratios, \( C_i/K_i, \frac{C}{K}, \) the individual and aggregate leisure times, \( l_i, l, \) average growth rate, \( \psi, \) volatility of the growth rate, \( \sigma_\psi, \) and one of the fiscal instruments given the other three policy parameters. Note that the tax and subsidy on the stochastic components of investment and the return to capital, have no effect on the equilibrium variables and thus \( \tau'_k = s' \) can be set arbitrarily.

Using (16a), (16d), and (16f), the macroeconomic equilibrium of the economy can be summarized by the following pair of equations that jointly determine the equilibrium mean growth rate, \( \psi, \) and average leisure \( l: \)

\[
\begin{align*}
\text{RR:} & \quad \psi = \frac{(1-\alpha)\Omega(l)(1-\tau_k)/(1-s)-\beta}{1-\gamma} - \frac{\gamma}{2} \Omega^2 \sigma^2, \\
\text{PP:} & \quad \psi = \Omega(l) \left[ 1 - \frac{\alpha l}{\eta} \frac{1}{1+\tau_c} \frac{l}{1-l} \right].
\end{align*}
\]

The first equation describes the relationship between \( \psi \) and \( l \) that ensures the equality between the risk-adjusted rate of return to capital and return to consumption. The second describes the combinations of the mean growth and leisure that ensure product market equilibrium holds.

2.4. The laissez-faire economy

It is convenient to examine the equilibrium in the absence of taxation. Setting all taxes and subsidies to zero, the equilibrium mean growth rate and leisure are determined by the following pair of equations:

\[
\text{RR:} \quad \psi = \frac{(1-\alpha)\Omega(l)-\beta}{1-\gamma} - \frac{\gamma}{2} \Omega(l)^2 \sigma^2,
\]
PP: \[ \psi = \Omega(l) \left( 1 - \frac{\alpha l}{\eta(1-l)} \right), \]

The laissez-faire RR and PP locuses are depicted in Figure 1, and their formal properties are derived in the Appendix.\(^{13}\) First, note that equation PP is always decreasing in \(l\), reflecting the fact that more leisure time reduces output, thus increasing the consumption-output ratio and having an adverse effect on the growth rate of capital. On the other hand, for RR we have

\[ \frac{\partial \psi}{\partial l} = \left( \frac{1-\alpha}{1-\gamma} - \gamma \Omega(l) \sigma^2 \right) \Omega'(l) \]

This expression is unambiguously negative for \(\gamma < 0\), as the empirical evidence suggests, and the case that we shall assume prevails. Intuitively, a higher fraction of time devoted to leisure reduces the productivity of capital, requiring a fall in the return to consumption. This is obtained if the growth of the marginal utility of consumption rises, that is, if the balanced growth rate falls. Under plausible conditions, the two schedules are concave, and an equilibrium exists if

\[ \alpha - \gamma + \frac{\beta}{A} > -\gamma \frac{1-\gamma}{2} A \sigma^2. \]

We will see in our numerical calibrations that this condition is met for reasonable parameter values.

3. The Determinants of the Distribution of Income

In order to examine the effect of risk on income distribution, we consider the expected relative income of an individual with capital \(k_i\). Her (expected) gross income is simply \(E(dY_i) = rK_i + wK(1-l_i)\), while expected average income is \(E(dY) = rK + wK(1-l)\). Using equation (15) to substitute for labor, we can express the relative (expected) income of individual \(i\), \(y_i \equiv E(dY_i)/E(dY)\), as

\[ y_i(l, k_i) = k_i + \frac{w}{(1+\eta)\Omega}(1-k_i) = k_i + \frac{\alpha}{(1+\eta)(1-l)}(1-k_i) \]

which we may write more compactly as:

\(^{13}\)See also Turnovsky (2000b).
Equation (18’) emphasizes that the distribution of income depends upon two factors, the initial (unchanging) distribution of capital, and the equilibrium allocation of time between labor and leisure, insofar as this determines factor rewards. The net effect of an increase in initial wealth on the relative income of agent $i$ is given by $\rho(l)$. As long as the equilibrium is one of positive growth, it is straightforward to show that

$$0 < \rho(l) < 1$$

Thus relative income is strictly increasing in $k_i$, indicating that although richer individuals choose a lower supply of labor, this effect is not strong enough to offset the impact of their higher capital income. As a consequence, the variability of income across the agents, $\sigma_y$, is less than their (unchanging) variability of capital, $\sigma_k$.

The second point to note is that we can rank different outcomes according to inequality without needing any information about the underlying distribution of capital. For a given distribution of capital, changes in risk or policy affect the distribution of income solely through their impact on relative prices, as captured by $\rho(l)$. Correia (1999) has shown that when agents differ only in their endowment of one good, there exists an ordering of outcomes by income inequality, as measured by second-order stochastic dominance.\footnote{Her results also require that the economy be amenable to Gorman aggregation, which is the case in our setup.} That ordering is determined by equilibrium prices, and is independent of the distribution of endowments.

To illustrate this, we consider an analytically convenient measure of inequality, the standard deviation of relative income, $\sigma_y$, where

$$\sigma_y = \rho(l)\sigma_k$$

Given the standard deviation of capital, $\sigma_k$, the standard deviation of income is a decreasing and

\footnote{Writing $\rho(l) = \frac{1}{(1+\eta)(1-\eta)} \left[ (\eta(1-l) - \alpha l) + (1-\alpha)(1-l) \right]$. If the equilibrium is one of positive growth, (17b) implies that the first term in brackets is positive, thus ensuring that $\rho(l) > 0$. The fact that $\rho(l) < 1$ is immediate from its definition.}
concave function of aggregate leisure time. This is because as leisure increases (and labor supply declines) the wage rate rises and the return to capital falls, compressing the range of income flows between the wealthy with large endowments of capital and the less well endowed.

The lower panel of Fig. 1 illustrates the DD locus, which can be shown to be concave in $l$. Thus, having determined the equilibrium allocation of labor from the upper panels in Fig. 1, (17c) determines the corresponding unique variability of income across agents.

Because taxes also have direct redistributive effects, we need to distinguish between the before-tax and after-tax distribution of income. We therefore define the agent’s after-tax (or net) relative income as

$$y^\text{NET}_i(l, k, \tau_k, \tau_w) \equiv \frac{r(1-\tau_k)k_i + w(1-\tau_w)(1-l_i)}{r(1-\tau_k) + w(1-\tau_w)(1-l)} = 1 - \rho^\text{NET}(l, \tau_w, \tau_k)(1-k_i)$$

(20a)

where, $\rho^\text{NET}$ summarizes the distribution of after-tax income and is related to corresponding before-tax measure, $\rho(l)$, by

$$\rho^\text{NET}(l, \tau_w, \tau_k) = \rho(l) + (1 - \rho(l))(1-\alpha)\frac{\tau_w - \tau_k}{\alpha(1-\tau_w) + (1-\alpha)(1-\tau_k)}$$

(20b)

with the standard deviation of after-tax income given by

$$\sigma^\text{NET}_y = \rho^\text{NET}(l, \tau_w, \tau_k)\sigma_k$$

(20c)

From (20a) and (20b) we see that fiscal policy exerts two effects on the after-tax income distribution. First, by influencing gross factor returns it influences the equilibrium supply of labor, $l$, and therefore the before-tax distribution of income, as summarized by $\rho(l)$. In addition, it has a direct redistributive effect, which is summarized by the second term on the right hand side of (20b). The dispersion of pre-tax income across agents will exceed the after-tax dispersion if and only if $\tau_k > \tau_w$.

As we will see below, in most cases tax increases affect the before-tax and after-tax distributions in opposite ways.

Lastly, we compute individual welfare. By definition, this equals the value function used to solve the intertemporal optimization problem evaluated along the equilibrium stochastic growth
path. For the constant elasticity utility function, the optimized level of utility for an agent starting from an initial stock of capital, \(K_{i,0}\), can be expressed as

\[
X(K_{i,0}) = \frac{1}{\gamma \beta - \gamma \left(\psi + 1/2(\gamma - 1)\sigma^2\right)} K_{i,0}^{\gamma}
\]  

(21)

The welfare of individual \(i\) relative to that of the individual with average wealth is then

\[
x(k_i) = \frac{(C_i/K_i)^{\gamma} l_i^{\eta} k_i^{\gamma}}{(C/K)^{\gamma} l^{\eta} k^{\gamma}} = \left(\frac{l_i}{l}\right)^{(1+\eta)\gamma} \frac{\eta}{1+\eta} \frac{1-k_i}{l},
\]

(22)

where the second term has been obtained by substituting for the consumption-capital ratio. Using equations (15), we can express relative welfare as

\[
x(k_i) = \left(k_i + \frac{\eta}{1+\eta} \frac{1-k_i}{l}\right)^{\gamma(1+\eta)}.
\]

(22')

Consider now two individuals having relative endowments \(k_2 > k_1\). Individual 2 will have both a higher mean income but also higher volatility. The transversality condition (13”) implies that if \(\gamma > 0\), then their relative welfare satisfies \(x(k_2) > x(k_1) > 0\), while if \(\gamma < 0\), \(x(k_1) > x(k_2) > 0\). However, in the latter case absolute welfare, as expressed by (19) is negative. Thus in either case, the better endowed agent will have the higher absolute level of welfare, so that the distribution of welfare moves together with that of income.

4. **The Relationship between Volatility and Inequality**

We now turn to the relationship between volatility, growth, and the distribution of income, focusing on how these relationships respond to an increase the volatility of production, \(\sigma^2\). In this section we examine the case of an economy without taxation. We discuss the relationship analytically and then supplement this with some numerical simulations.

4.1. **Analytical properties**

The effect of risk operates through its impact on the incentives to accumulate capital. An
increase in $\sigma^2$ shifts the RR curve only, and for $\gamma < 0$, it shifts the RR curve upwards, as seen in Fig. 2. Given the fraction of time devoted to leisure, the shift in RR tends to increase the growth rate. The higher $\psi$ increases the return to consumption, which raises the labor supply, and hence the return to capital relative to that of consumption, causing a further increase in the growth rate. In addition greater risk increases the variance of the growth rate, $\sigma^2_{\psi} = \Omega^2 \sigma^2$, because of both the direct effect of $\sigma^2$ and the indirect impact of a lower $l$ on $\Omega$.

From equation (18’) the effect of an increase in risk on the relative gross income of an agent with capital share $k_i$ is given by

$$\text{sgn} \left( \frac{dy(k_i)}{d\sigma^2} \right) = -\text{sgn} \left( \frac{\partial \rho}{\partial l} \frac{dl}{d\sigma^2}(1-k_i) \right) = \text{sgn}(k_i - 1)$$

An increase in the average leisure time raises the income share for those with a wealth share below the average, and reduces the income share of those with wealth above average. That is, for a given distribution of wealth, a higher equilibrium value of $l$ reduces the relative income of individuals with wealth (and hence income) below the mean, and increases the relative income of agents with an above-average endowment. Consequently, inequality rises. Greater volatility of the production shock, as measured by a higher $\sigma^2$, increases the average growth rate and its volatility, reduces the amount of time devoted to leisure, and increases income inequality.

The intuition for these results is straightforward. Because agents are sufficiently risk-averse, a greater variance of output has a strong income effect that makes them increase savings. Consequently, the growth rate increases. Note from the PP locus that the allocation of labor is unaffected by $\sigma^2$ for a given growth rate. A higher growth rate, however, implies higher future wages, and hence higher consumption for any extra time spent at work. It therefore reduces leisure and the change in leisure time, in turn, affects the distribution of income. An increase in leisure time reduces the labor supply, lowering the return to capital and increasing that to labor. Since labor is more equally distributed than capital, the income gap between any two individuals falls. Note that with an inelastic supply of labor, risk would not affect relative incomes. In this case, the income of agent $i$ would be given by $y_i = (w + rk_i)/(w + r)$. With the AK technology resulting in a constant wage and interest rate, this expression would be unaffected by risk. In our setup risk matters because
it affects the growth rate, and this, in turn, impacts the labor supply and factor rewards.

4.2. **Calibrations**

To obtain further insights into the impact of risk on the equilibrium, and in particular the relationship between growth and income inequality, we perform some numerical analysis. We calibrate a benchmark economy, using the following mostly conventional parameter values:

**Calibration Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production</td>
<td>( A = 0.33 ), ( \alpha = 0.33 )</td>
</tr>
<tr>
<td>Taste</td>
<td>( \beta = 0.04 ), ( \gamma = -2 ), ( \eta = 1.25 )</td>
</tr>
<tr>
<td>Risk</td>
<td>( \sigma = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5 )</td>
</tr>
</tbody>
</table>

The choice of production elasticity of labor measured in efficiency units implies that 33% of output accrues to labor. The estimated labor share of output in industrial economies is usually much higher, around 60%. Such a figure presents two problems. First, one consequence of the Romer technology being assumed here, is that such a value implies an implausibly large externality from aggregate capital. Moreover, such a high value of \( \alpha \) would mean that the elasticity of firm output with respect to aggregate capital is greater than with respect to own capital. Second, empirical studies using a “broad” concept of capital that includes investment in human capital obtain a much lower output elasticity of labor than that implied by labor share data. Hence, in line with the estimations of Mankiw, Romer, and Weil (1992) we use a value of 33%. The choice of the scale parameter \( A = 0.33 \), is set to yield a plausible value for the equilibrium capital-output ratio.

Turning to the taste parameters, the rate of time preference of 4% is standard, while the choice of the elasticity on leisure, \( \eta = 1.25 \), implies that about 75% of time is devoted to leisure, consistent with empirical evidence. Estimates of the coefficient of relative risk aversion are more variable throughout the literature. Values of the order of \( \gamma = -18 \) (and larger) have sometimes been assumed to deal with the equity premium puzzle (see Obstfeld, 1994). However, these tend to yield implausibly low values of the equilibrium growth rate. By contrast, real business cycle theorists
routinely work with logarithmic utility functions ($\gamma = 0$). More recently, a consensus seems to be emerging of values between 2 and 5 (see Constantinides, Donaldson, and Mehra 2002) and our choice of $\gamma = -2$ (coefficient of relative risk aversion = 3) is well within that range.

Our main focus is on considering increases in exogenous production risk, which we let vary between $\sigma = 0.05$ and $\sigma = 0.5$. The value $\sigma = 0.05$ is close to the mean for OECD countries considered by Gali (1994) and Gavin and Hausmann (1995). Gavin and Hausmann present estimates for a wide range of countries and $\sigma = 0.10$ corresponds to countries subject to medium production risk. The highest value in our calibrations has been chosen so as to yield a standard deviation of the rate of growth of around 10%, in line with the estimates for high-volatility economies (see Ramey and Ramey, 1995).

Table 1 reports our assumed as well as some actual distributions. The first line in Table 1 reports the distribution of wealth for the US. In 1998, the bottom 40% of the population held 0.2% of total wealth, while the top 20% owed 83.8% of the total. The second line gives our assumed distribution. The next two lines are the distributions of income in the US in 1997 and in Venezuela – a high inequality, high volatility economy- in 1998. The last line reports the income distribution generated by the model, using the hypothetical wealth distribution for the case of low risk $\sigma = 0.05$. To obtain it we have assumed that the bottom income group has no wealth and no labor endowment (i.e, zero income). Otherwise, since wages are identical for all workers, we would have a very large group with the same income at the bottom of the income distribution. Our assumption implies that the income share of the two bottom groups is 13.99 and hence of a similar magnitude to that observed in the data. The resulting Gini coefficient, 42.89, is close to recent estimates for the US.

Table 2 reports the impact of increases in the volatility of the output shock on the equilibrium labor supply, the average rate of growth and its standard deviation, the Gini coefficient of income, and on overall welfare. Welfare changes reported are calculated as the percentage equivalent variations in the initial stock of capital of the average individual necessary to maintain the level of utility following the increase in risk from the benchmark level $\sigma = 0.05$ reported in the first row.

Line 1 of the table suggests that treating $\sigma = 0.05$ as a benchmark case leads to a plausible

---

equilibrium, having a 3.28% mean growth rate and 1% relative standard deviation, with 76% of time allocated to leisure. The implied distribution of income is also plausible, as noted. As risk increases from $\sigma = 0.05$, Table 2 indicates the following. The mean growth rate increases, as does its standard deviation. The net effect of the greater risk dominates the positive effect of the higher growth rate, so that the increase in risk reduces average welfare. It should be noted that for the plausible range of $\sigma < 0.20$, the welfare loss is relatively modest. This is a characteristic limitation of this class of model having only aggregate risk, and has been discussed elsewhere in the literature. More to the point here, we see that greater risk is associated with a substitution toward more labor (less leisure), and an increase in income inequality -- as measured by the Gini coefficient -- consistent with the formal analysis presented in Section 4.1.

In terms of magnitudes, the effect of risk on the Gini coefficient is quite modest, at least for plausible degrees of risk, with an increase in risk from $\sigma = 0.05$ to $\sigma = 0.5$ raising the Gini coefficient by 1.5 points. It is interesting to note that income inequality in the US increased by 2.5 Gini points between 1980 and 1990, and that this has been considered a sizeable increase. Moreover, small changes in risk may play a significant role if they give rise to large policy responses.

5. **Taxation**

A familiar feature of the Romer (1986) model is that by increasing the growth rate, an investment subsidy may move the equilibrium closer to the social optimum. When agents are heterogeneous, two questions arise. First, how to finance this subsidy if the government is concerned about inequality as well as about average welfare. An investment subsidy raises the return to capital and will tend to favor those with large capital holdings. If the subsidy were financed by a lump-sum tax, the system would redistribute away from those with lower incomes to those with higher incomes. Are there ways in which this reverse redistribution can be avoided? Second, we want to know whether the use of first-best policies has any implication for the relationship between volatility and inequality. In this section we investigate these questions in some detail. We begin by deriving

17 Most notably it is characteristic of Lucas’s (1987) of the cost of business cycles, and it is also discussed at more length for a model closer to present by Turnovsky (2000b).
the first-best optimal rate of growth and allocation of labor.

5.1. The first-best optimum

Given the externality stemming from the aggregate capital stock, finding the first-best optimums amounts to solving the following problem:

$$\max E_0 \int_0^\infty \frac{1}{\gamma} \left( C_i(t) l^\eta \right)^\gamma e^{-\beta t} dt,$$

subject to

$$dK_i = (\Omega K_i - C_i) dt + \Omega K_i du$$

In the Appendix we show that the solution to this problem is given by the equations

$$\psi' = \frac{\Omega(i) - \beta}{1 - \gamma} - \frac{\gamma}{2} \Omega(i)^2 \sigma^2$$

$$\psi = \Omega(i) \left[ 1 - \frac{\alpha}{\eta} \frac{i}{1 - i} \right]$$

$$\sigma_y = \rho(i) \sigma_\epsilon$$

$$\frac{C}{K} = \frac{\alpha \Omega(i) i}{\eta} \frac{i}{1 - i}$$

where the tilde denotes the first-best optimum. Note that the only difference with the solution to the competitive equilibrium in the absence of taxes is that the social rate of return to capital now takes into account the production externality and hence exceeds the private return. The \(RR'\) schedule lies above \(RR\). Given that the \(PP\) schedule is steeper than \(RR\), the upward shift of \(RR\) results in a higher growth rate, lower leisure, and therefore increases inequality, as can be seen from Figure 2.19

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18 The transversality condition (12) for the central planner’s problem again reduces to (13) but is now automatically satisfied without further restrictions being imposed.

19 The reason why the social planner chooses less leisure is that there are in fact two externalities in the model. On the one hand, a greater individual stock of capital increases the aggregate level of technology. On the other, a higher labor supply raises the marginal product of capital and induces greater accumulation of capital, thus increasing the level of technology.
5.2 First-best taxation

Comparing the first-best optimum, described by R’R’, and PP with the decentralized equilibrium, RR, PP we can see that the tax-subsidy system can be used to attain the optimal growth rate, leisure time, and consumption/capital ratio by setting

\[ \frac{1 - \tau_w}{1 + \tau_c} = 1, \quad \text{i.e.} \quad \tau_w = -\tau_c, \quad (24a) \]

\[ (1 - \alpha) \frac{1 - \tau_k}{1 - s} = 1, \quad \text{i.e.} \quad s = \alpha + (1 - \alpha) \tau_k, \quad (24b) \]

\[ \frac{1 - \tau_k}{1 - \tau_w} = \frac{1 - \eta}{1 - \alpha} \left[ 1 - \frac{1 - \bar{l}}{l} \right], \quad (24c) \]

where the last equation is obtained from the government’s budget constraint, (16g). The first two equations represent intuitive optimality conditions. The first states that any wage tax should be offset with an equivalent consumption tax so as not to distort the leisure-consumption choice. Interpreting the tax on wage income as a negative tax on leisure, (24a) states that the two utility enhancing goods, consumption and leisure, should be taxed uniformly. This result is a straightforward application of the Ramsey principle of optimal taxation; see Deaton (1981) and Lucas and Stokey (1983). If the utility function is multiplicatively separable in \( c \) and \( l \), as it is here, then the uniform taxation of leisure and consumption is optimal. The second condition simply ensures that the private rate of return on investment must equal the social return, and for this to be so the subsidy to investment must exceed the externality by an amount that reflects any tax on capital income.\(^{20}\) Note from the third equation that unless consumption equals total income, (in which case there is zero growth), the replication of the first optimum requires differential taxes on wages and capital; \( \tau_w \lesssim \tau_k \), according to whether there is positive or negative growth.

From (24a) – (24c) we obtain the following solutions for the tax rates, given an arbitrarily chosen subsidy rate, \( s \):

\(^{20}\)The optimal tax rates set out in (24) are similar to those obtained by Turnovsky (2000a) in a pure deterministic representative agent endogenous growth model.
\[ \hat{\tau}_k = \frac{s - \alpha}{1 - \alpha} \]
\[ \hat{\tau}_w = \frac{\eta(1 - \bar{I}) - \bar{I}s}{\eta(1 - l) - \bar{I}} = -\hat{\tau}_c \]

There are two things to note about these expressions. First, \( \hat{\tau}_k \) is very sensitive to the (arbitrary) choice of \( s \). Second, it is possible for the wage tax to be positive and the consumption tax negative, and vice versa, and as we will see below, both situations arise in our calibration results.

What is the impact of the first-best taxation system on distribution? Recall that the dispersion of gross income is given by (17c), where \( \rho(l) \) is a decreasing function of leisure time. Since the policy increases the time allocated to labor, it will increase gross income inequality. The dispersion of net income in the decentralized economy that mimics the centrally planned equilibrium is obtained by substituting the tax rates, (25a), (25b), into (20b) to yield

\[ \rho^{\text{NET}}(l; \tau_w, \tau_k) = 1 - \frac{\alpha}{(1 + \eta)(1 - l) \left( 1 + \alpha - \eta \left( (1 - l)/l \right) \right)} \]

The striking aspect about (26) is that the distribution of net income is independent of the (arbitrary) choice of fiscal instruments employed to achieve this objective. As long as the equilibrium is one with positive growth, the optimal tax requires \( \hat{\tau}_w < \hat{\tau}_k \). Then \( \rho^{\text{NET}} < \rho \), and net income is less dispersed than is gross income. In addition, in all of our simulations we find that the direct redistributive effect of taxation dominates the indirect effect of changes in factor prices so that the distribution of net income is less unequal than in the economy without taxes.

When the first-best tax system is implemented, the effect of risk on growth and leisure is equivalent to that in the laissez-faire economy, as can be easily verified from equations (17). Greater risk is therefore associated with a greater supply of labor and hence with more pre-tax inequality. The effect of risk on after-tax inequality is, however, ambiguous. Differentiating (26) with respect to \( l \), we can see that there are two opposing effects. On the one hand, more leisure tends to reduce pre-tax inequality. On the other, and as long as the subsidy rate is less than 1, a lower labor supply implies that a higher wage tax is required in order to finance any given subsidy rate (see (25b)), making the fiscal system less progressive. Either effect can dominate, implying that greater risk need
not result in a more unequal distribution of after-tax income.

5.3 Calibrations

Table 3a reports the numerical effects of implementing the first-best taxes using the parameter values described in section 4.2. It offers a number of insights that both reinforce and complement our analytical results. First, we see that the policy involves a substantial reduction in leisure time (around 4 per cent), raising the growth rate by 2.5 percentage points, and only slightly increasing volatility. \( \Delta X \) is the increase in the welfare of the average individual, measured as the percentage variation over that in an economy with the same level of risk in the benchmark equilibria (i.e. those in Table 2). First-best taxation increases the welfare of the average individual in the economy by over 4%.

The effects on income distribution are substantial. The large reduction in leisure time results in a large increase in pre-tax inequality, of about 3.5 percentage points. However, the redistributive effect is strong enough to offset this effect and yield an overall reduction in the Gini coefficient of net income. Gross and net income inequality move in opposite ways. The extent of the reduction in post-tax inequality relative to the economy without subsidies depends heavily on the degree of risk in the economy. For the lowest risk level, the change is small, but for \( \sigma = 0.5 \) it amounts to almost two Gini points. The reason for this is that since greater risk is associated with a greater labor supply, and hence a greater wage bill, the wage tax required to finance a give subsidy is lower, and hence the fiscal system is more progressive.

Table 3a also illustrates the analytical results that the first-best equilibrium can be replicated by a variety of tax/subsidy configurations, each of which leads to precisely the same post-tax distribution of income. The sensitivity of the tax regime to changes in the subsidy rate is also borne out. Consider for example the case of low risk, \( \sigma = 0.05 \). In the absence of a subsidy to investment, the first best equilibrium can be sustained if income form capital and labor are subsidized at the rates of 49% and 83%, respectively, while consumption is taxed at 83%! This is hardly a politically viable tax structure. But the first-best equilibrium can also be attained if, more reasonably,

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21 The implied percentage increase in labor supply is much larger, being of the order of 20%.
investment and consumption are subsidized at 60% and 27% respectively, with taxes on labor income and capital income of 27% and 40%, respectively.

Table 3b replicates the results in table 3a with a lower coefficient of relative risk aversion ($\gamma = -1.15$, rather than $-2$). The table shows how, for certain parameter values, the positive relationship between volatility and inequality may disappear when the first-best tax system is implemented. As is always the case, greater volatility is associated with greater pre-tax inequality, but in this particular example the distribution of after-tax income is (virtually) unchanged by the degree of risk. Our numerical results highlight the fact that the divergence between pre- and post-tax inequality is greater the more risky the economy is, implying that high risk economies face a trade-off between pre- and post-tax inequality.

6. **Conclusions**

Stochastic shocks are a major source of income disparities, and an extensive literature has explored how “luck” and the market’s tendency towards convergence combine to create persistent inequality. Yet this literature is concerned with idiosyncratic shocks that have no relation with aggregate shocks. The idea that aggregate uncertainty may also affect the distribution of income remains to be explored, and this paper is a first step in that direction.

We have used an AK stochastic growth model to show that, when agents differ in their initial stocks of capital, greater growth volatility is associated with a more unequal distribution of income. Greater risk tends to increase the supply of labor, reducing wages and raising the interest rate. If capital endowments are unequally distributed, while labor endowments are not, the change in factor prices raises the return to the factor that is the source of inequality, and the distribution of income becomes more spread.

The endogeneity of the labor supply also implies that policies aimed at increasing the growth rate will have distributional implications, and we have examined how these differ depending on the particular form of the policies. In particular, we have compared financing an investment subsidy through a capital income tax, a wage tax, or a consumption tax.

Our analysis yields two main conclusions. First, it is possible to simultaneously increase the
growth rate and reduce net income inequality. In many instances, we find that policies that generate faster growth are associated with a reduction in the Gini coefficient, allowing the policymaker to attain both efficiency and equity goals.

Second, it is often the case that fiscal policy has opposite effects on the distribution of gross and net income. These results highlight the fact that rather than the usual tradeoff between equity and efficiency, policymakers concerned with the distribution of income may face a tradeoff between pre- and post-tax inequality. Moreover, the divergence between pre and post tax inequality is greater the more risky the economy is. Understanding which type of inequalities agents and the social planner care about becomes essential, particularly in high risk economies, and it raises the question of whether a slightly more unequal distribution of both gross and net incomes may, in certain cases, be a more viable policy than a huge, but offset, increase in pre-tax inequality.
Fig 1: Equilibrium Growth, Employment, and Income Distribution
Fig 2: Increase in Production Risk

\[ \frac{-\beta}{1-\gamma} \]
Table 1
The Distribution of Income and Wealth

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Gini</th>
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</thead>
<tbody>
<tr>
<td>US: wealth shares</td>
<td>0.2</td>
<td>4.5</td>
<td>11.9</td>
<td>83.4</td>
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<tr>
<td>Assumed wealth shares</td>
<td>0</td>
<td>0</td>
<td>4.0</td>
<td>12.0</td>
<td>84.0</td>
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<tr>
<td>US: income shares</td>
<td>5.2</td>
<td>10.5</td>
<td>15.6</td>
<td>22.4</td>
<td>46.4</td>
<td>40.8</td>
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<tr>
<td>Venezuela: income shares</td>
<td>3.0</td>
<td>8.2</td>
<td>13.8</td>
<td>21.8</td>
<td>53.2</td>
<td>49.5</td>
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<tr>
<td>Simulated income shares (σ = 0.05)</td>
<td>0</td>
<td>13.99</td>
<td>15.76</td>
<td>19.28</td>
<td>50.97</td>
<td>42.89</td>
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Table 2
Growth, Volatility, and the Distribution of Income

<table>
<thead>
<tr>
<th></th>
<th>l</th>
<th>ψ</th>
<th>σν</th>
<th>ΔX</th>
<th>Gini(y)</th>
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<td>σ = 0.05</td>
<td>76.11</td>
<td>3.28</td>
<td>1.03</td>
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<tr>
<td>σ = 0.1</td>
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<td>2.06</td>
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<tr>
<td>σ = 0.2</td>
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<tr>
<td>σ = 0.3</td>
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<td>6.21</td>
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<tr>
<td>σ = 0.4</td>
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<td>4.00</td>
<td>8.31</td>
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<td>43.89</td>
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<tr>
<td>σ = 0.5</td>
<td>74.91</td>
<td>4.43</td>
<td>10.45</td>
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<td>44.46</td>
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Table 3a
First-best Taxation: baseline parameters

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<tr>
<th>$\sigma$</th>
<th>$s$</th>
<th>$\tau_w$</th>
<th>$\tau_k$</th>
<th>$l$</th>
<th>$\psi$</th>
<th>$\sigma_\psi$</th>
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<th>Gini(yny)</th>
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<td>0.05</td>
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Appendix

This appendix provides some of the technical details underlying the derivations of the equilibrium conditions (8) and (16a) to (16g).

A.1 Consumer optimization

Agent $i$’s stochastic maximization problem is to choose her individual consumption-capital ratio and the fraction of time devoted to leisure to maximize expected lifetime utility

$$\max E_0 \int_0^\infty \frac{1}{\gamma} \left(C_i(t)^\gamma\right) e^{-\beta t} dt, \quad -\infty < \gamma < 1, \eta > 0, \eta \eta < 1 \quad (A.1a)$$

subject to her individual capital accumulation constraint

$$dK_i = \frac{(1-\tau_k) r K_i + (1-\tau_w) w(1-l_i) K - (1+\tau_e) C}{1-s} dt + K_i dk \quad (A.1b)$$

and the aggregate capital accumulation constraint

$$dK = \frac{(1-\tau_k) r K + (1-\tau_w) w(1-l) K - (1+\tau_e) C}{1-s} dt + K dk \quad (A.1b')$$

together with the economy-wide shock

$$dk = \frac{1-\tau_k'}{1-s'} \Omega du. \quad \quad (A.1c)$$

Since the agent perceives two state variables, $K_i, K$, we consider a value function of the form

$$V(K_i, K, t) = e^{-\beta t} X(K_i, K)$$

the differential generator of which is

$$\Psi[V(K_i, K, t)] = \frac{\partial V}{\partial t} + \left[\frac{1-\tau_k}{1-s} r K_i - \frac{1+\tau_e C_i}{1-s} K_i + w(1-l) \frac{1-\tau_w}{1-s} K \right] V_{K_i}$$

$$+ \left(\frac{1-\tau_k}{1-s} r - \frac{1+\tau_e C}{1-s} K + w(1-l) \frac{1-\tau_w}{1-s} K \right) K V_K + \frac{1}{2} \sigma_k^2 K_{K,k_i}^2 + \sigma_{K,K} K_i K V_{K,k_i} + \frac{1}{2} \sigma_k^2 K^2 V_{K_K} \quad (A.2)$$
The individual’s problem is to choose consumption, leisure, and the rate of capital accumulation to maximize the Lagrangian

\[ e^{-\beta t} \frac{1}{\gamma} \left( C_i^{\eta} \right)^\gamma + \Psi \left[ e^{-\beta t} X(K_i, K) \right]. \] (A.3)

In doing this, she takes the evolution of the aggregate variables and the externality as given. Taking the partial derivatives with respect to \( C_i \) and \( l_i \), and cancelling \( e^{-\beta t} \) yields

\[ \frac{1}{C_i} \left( C_i^{\eta} \right)^\gamma = \frac{1 + \tau_c}{1 - s} X_{K_i} \] (A.4a)

\[ \frac{\eta}{l} \left( C_i^{\eta} \right)^\gamma = \frac{1 - \tau_w}{1 - s} wKX_{K_i} \] (A.4b)

In addition, the value function must satisfy the Bellman equation

\[ \max \left\{ e^{-\beta t} \frac{1}{\gamma} \left( C_i^{\eta} \right)^\gamma + \Psi \left[ e^{-\beta t} X(K_i, K) \right] \right\} = 0 \] (A.5)

The Bellman equation is a function of two state variables, individual and aggregate capital, and hence it is a partial differential equation in these two variables. Using equations (A.1b) and (A.1b’), and given (A.2), the Bellman equation can be written as

\[ \frac{1}{\gamma} \left( C_i^{\eta} \right)^\gamma - \beta X(K_i, K) + \frac{E(dK_i)}{dt} X_{K_i} + \frac{E(dK)}{dt} X_K \\
+ \frac{1}{2} \frac{E(dK_i)^2}{dt} X_{K_i,K_i} + \frac{E(dK_i)dK}{dt} X_{K_i,K} + \frac{1}{2} \frac{E(dK)^2}{dt} X_{K,K} = 0 \] (A.6)

Next we take the partial derivative of the Bellman equation with respect to \( K_i \), noting that \( l_i \) is independent of \( K_i \), while \( C_i \) is a function of \( K_i \) through the first-order condition (A.4a),

\[ \frac{1}{C_i} \left( C_i^{\eta} \right)^\gamma C_{i,K_i} - \beta X_{K_i} + \frac{E(dK_i)}{dt} X_{K_i,K_i} + \left[ r \frac{1 - \tau_k}{1 - s} - C_{i,K_i} \right] X_{K_i} + \frac{E(dK)}{dt} X_{K_i,K} + K_iX_{K_i,K_i} \sigma^2_K \\
+ \frac{1}{2} \frac{E(dK_i)^2}{dt} X_{K_i,K_i} + \frac{E(dK_i)dK}{dt} X_{K_i,K} + KX_{K,K} \sigma_{K,K} + \frac{1}{2} \frac{E(dK)^2}{dt} X_{K,K} = 0 \] (A.7)

Consider now \( X_{K_i} = X_{K_i}(K_i,K) \). Taking the stochastic differential of this quantity yields:
\[ dX_k = X_{K,K} dK_i + X_{K,K} dK + \frac{1}{2} X_{K,K,K} (dK_i)^2 + X_{K,K,K} (dK_i)(dK) + \frac{1}{2} X_{K,K,K} (dK)^2 \]  \hspace{1cm} (A.8)

Taking expected values of this expression, dividing by \( dt \), and substituting the resulting equation along with (A.4a) into (A.7) leads to:

\[
\begin{bmatrix}
  r \frac{1-\tau_k}{1-s} - \beta
\end{bmatrix} X_{K_i} + \left[ K_i X_{K,K} + KX_{K,K} \right] \sigma_k^2 + \frac{E(dX_k)}{dt} = 0 , \hspace{1cm} (A.9)
\]

The solution to this equation is by trial and error. Given the form of the objective function, we propose a value function of the form:

\[ X(K_i, K) = cK_i^{\gamma - \gamma_2} K^{\gamma_2} \]  \hspace{1cm} (A.10)

where the parameters \( c, \gamma_2 \) are to be determined. From (A.10) we obtain:

\[
\begin{align*}
X_{K_i} &= (\gamma - \gamma_2)X/K_i; \quad X_K = \gamma_2 X/K; \\
X_{K,K_i} &= (\gamma - \gamma_2)(\gamma - \gamma_2 - 1)X/K_i^2; \\
X_{K,K} &= (\gamma - \gamma_2)\gamma_2 X/KK; \quad X_{KK} = (\gamma_2 - 1)\gamma_2 X/K^2. 
\end{align*} \hspace{1cm} (A.11)
\]

We can now use equation (A.12) to re-express (A.11) as

\[
\frac{E(dX_k)}{X_{K_i} dt} = \beta - r \frac{1-\tau_k}{1-s} + (\gamma - 1)\sigma_k^2 \]  \hspace{1cm} (A.12)

Now, returning to the first-order condition (A.4a), computing the stochastic differential of this relationship and taking expected values yields

\[
\frac{E\left(dX_k\right)}{X_{K_i}} = (\gamma - 1) \frac{E(dC_i)}{C_i} + \frac{1}{2} (\gamma - 1)(\gamma - 2)E\left(\frac{dC_i}{C_i}\right)^2 \]  \hspace{1cm} (A.13)

Along the balanced growth path, \( C_i/K_i \) is constant. Hence \( dC_i/C_i = dK_i/K_i = \psi dt + dw \), and thus

\[
\frac{E\left(dX_k\right)}{X_{K_i} dt} = (\gamma - 1)\psi + \frac{1}{2} (\gamma - 1)(\gamma - 2)\sigma_k^2 \]  \hspace{1cm} (A.14)

As will be shown below (see equation (A.26)), the government’s balanced budget implies that the stochastic component of the individual budget constraint is
\[ dk = \Omega du. \]  

Combining (A.13), (A.14), and (A.15) yields the mean growth rate of individual consumption

\[ \psi = \frac{r ((1-\tau_k)/(1-s)) - \beta}{1-\gamma} - \frac{\gamma \Omega^2 \sigma^2}{2}. \]  

The labor supply is obtained from the first-order conditions (A.4a) and (A.4b), namely

\[ C_i = \frac{w(1-\tau_w)}{\eta (1+\tau_c)} K_l. \]  

Dividing (A.17) by \( K \) we obtain the individual consumption to wealth ratio,

\[ \frac{C_i}{K_i} = \frac{w(1-\tau_w) l_i}{\eta (1+\tau_c) k_i}, \]  

and summing over all agents we have the aggregate consumption to wealth ratio,

\[ \frac{C}{K} = \frac{w(1-\tau_w) l}{\eta (1+\tau_c)}. \]  

From the individual budget constraint, the rate of growth is

\[ \psi = \frac{r (1-\tau_k) + w (1-\tau_w) 1-l_i}{1-s} \frac{1+\tau_c C_i}{1-s K_i}, \]  

which using (A.18) and rearranging gives

\[ 1-l_i = \frac{1}{1+\eta} - \frac{\eta r (1-\tau_k) / (1-s) - \psi w (1-s)}{1-\tau_w k_i}. \]  

**A.2 The government’s budget constraint**

The balanced budget implies

\[ sE(dK)dt + s'Kdk = [\tau_c C + \tau_k r_k K + \tau_w w (1-l) K] dt + \tau'_k Kdu. \]  

Equating the deterministic and the stochastic components of (A.22) yields :
\[ \tau_k' d\kappa = s'dk , \quad (A.23) \]

\[ \tau_k r_k + \tau_c w(1-l) + \tau_c \frac{C}{K} = s \psi . \quad (A.24) \]

Note that (A.1c) implies
\[ \tau_k' = s' . \quad (A.25) \]

As a result, the stochastic component of individual capital accumulation is
\[ d\kappa = \Omega d\kappa . \quad (A.26) \]

### A.3 Macroeconomic equilibrium

Note that the growth rate is the same for all agents, irrespective of their initial wealth holdings. Equation (A.16) is hence the mean growth rate of the economy. The dynamic evolution of the aggregate stock of capital is given by
\[ \frac{dK}{K} = \left[ r \left( \frac{(1-\tau_k)/(1-s)}{1-\gamma} - \beta \frac{\gamma}{2} \Omega^2 \sigma^2 \right) - \frac{\psi}{\Omega} \right] dt + \Omega d\kappa \]

and the standard deviation (volatility) of the growth rate is
\[ \sigma_\psi = \Omega \sigma . \quad (A.27) \]

Summing (A.21) over all agents gives a relationship between the aggregate labor supply and the growth rate,
\[ 1-l = \frac{1}{1+\eta} - \eta \left( \frac{\phi-\psi}{1+\eta} \right) - \frac{1-s}{w} \left( 1-\tau_k \right) . \quad (A.28) \]

Goods market equilibrium requires \( dK = \Omega K (dt+du) - C dt \), which taking expectations and dividing by \( K \) yields,
\[ \psi = \Omega - \frac{C}{K} . \quad (A.29) \]

Equations (A.16), (A.27) (A.18), (A.19), (A.20), (A.29), and (A.24) are the macroeconomic...
equilibrium conditions as specified in equations (16a) – (16g), respectively

In the absence of taxation, the equilibrium reduces to

\[
\frac{C_i}{K_i} = \frac{w l_i}{\eta k_i}, \quad (A.30a)
\]

\[
\psi = \frac{r - \beta - \frac{\gamma}{2} \Omega^2 \sigma^2}{1 - \gamma}, \quad (A.30b)
\]

\[
\psi = r + w(1 - l) - \frac{C_i}{K_i}, \quad (A.30c)
\]

\[
\frac{C}{K} = \frac{w l}{\eta}, \quad (A.30d)
\]

\[
\sigma_\psi = \Omega \sigma, \quad (A.30e)
\]

which jointly determine the consumption-capital ratio, the average growth rate, the labor supply, and the volatility of growth.

A.4 Existence of a balanced growth equilibrium

It suffices to focus on the economy without taxation; the introduction of taxes leads to minor modifications and can be analyzed analogously. Differentiating the relations in (A.30) we obtain

\[
\frac{\partial \psi}{\partial l} \bigg|_{RR} = -\frac{\alpha \Omega(l)}{1 - l} \left(1 - \frac{\alpha}{1 - \gamma} - \frac{\gamma}{2} \Omega(l) \sigma^2\right) < 0, \quad (A.31a)
\]

\[
\frac{\partial \psi}{\partial l} \bigg|_{PP} = -\frac{\alpha \Omega(l)}{\eta(1 - l) \left(1 + \eta + (1 - \alpha) \frac{l}{1 - l}\right)} < 0, \quad (A.31b)
\]

so that both schedules have a negative slope. Using the fact that \( \Omega = A(1 - l)^\alpha \), and under the assumption that \( \alpha < 1/2 \), both the \((PP)\) and \((RR)\) schedules can be shown to be strictly concave (see Turnovsky, 2000b, for more details). Also

\[
\psi_{RR}(l = 0) = \frac{A(1 - \alpha) - \beta}{1 - \gamma} - \frac{\gamma}{2} A^2 \sigma^2, \quad \psi_{RR}(l = 1) = -\frac{\beta}{1 - \gamma}
\]

\[
\psi_{PP}(l = 0) = A, \quad \psi_{PP}(l = 1) = -\infty.
\]
A necessary and sufficient condition for the existence of a unique equilibrium is \( \psi_{PP}(l = 0) > \psi_{RR}(l = 0) \). In this case, the \((PP)\) schedule is below \((RR)\) for \( l = 1 \), and the two schedules only cross once. This condition is satisfied when

\[
\alpha - \gamma + \frac{\beta}{A} > -\gamma \frac{1-\gamma}{2} A\sigma^2, \tag{A.32}
\]

i.e. when risk is not excessively high. When equation (A.32) is not satisfied either an equilibrium does not exists or there are two.

Note also that the \(PP\) schedule is steeper than \(RR\) if and only if

\[
\frac{1+\eta}{\eta} \left( \frac{1-\alpha}{1-l} \right) > \frac{1-\alpha}{1-\gamma} \Omega \sigma^2. \tag{A.33}
\]

Since at \( l = 0, \ l/(1- l) \) has its lowest and \( \Omega(l) \) its highest possible value, \(PP\) is everywhere steeper than \(RR\) if and only if

\[
\frac{1+\eta}{\eta} \left( \frac{1-\alpha}{1-\gamma} > -\gamma \sigma^2. \right. \tag{A.34}
\]

### A.5 The centrally planned economy

The social planner’s problem (23), leads to the following Bellman equation

\[
\frac{1}{\gamma} \left( C_i^{\eta n} \right)^\gamma - \beta X(K_i, K) + \frac{E(\Omega K_i - C_i)}{dK} X_{K_i} + \frac{1}{2} \frac{E(dK_i)^2}{dt} X_{K_i K_i} = 0. \tag{A.6'}
\]

Taking the partial derivative of this equation with respect to \( K_i \) then yields

\[
\frac{1}{C_i} \left( C_i^{\eta n} \right)^\gamma C_{i,K_i} - \beta X_{K_i} + \frac{E(dK_i)}{dt} X_{K_i K_i} + (\Omega - C_{i,K_i}) X_{K_i} + K_i X_{K_i K_i} \sigma_k^2 + \frac{1}{2} \frac{E(dK_i)^2}{dt} X_{K_i K_i K_i} = 0 \tag{A.7'}
\]

and hence

\[
\frac{E(dX_{K_i})}{X_{K_i} dt} = -(\Omega - \beta) + (\gamma - 1)\sigma_k^2, \tag{A.13'}
\]
which together with (A.14) and (A.15) above yield (16a').

The first order conditions with respect to consumption and leisure, (16), together imply

\[ \frac{C_i}{K_i} = \frac{wl_i}{\eta}, \]  

(A.19')

Goods market equilibrium is again given by equation (A.29). Using (A.19'), the equilibrium conditions can be expressed as

\[ \psi = \frac{\Omega(l) - \beta}{1 - \gamma} - \frac{\gamma}{2} \Omega(l)^2 \sigma^2, \]  

(A.35)

\[ \psi = \Omega(l) - \frac{C}{K} = \Omega(l) \left[ 1 - \frac{\alpha}{\eta} \frac{l}{1 - l} \right]. \]  

(A.36)

(A.36) is strictly decreasing and concave in \( l \). Differentiating (A.35), we obtain

\[
\frac{\partial \psi}{\partial l} \bigg|_{RR'} = -\frac{\Omega(l)}{1 - l} \left[ \frac{1}{1 - \gamma} - \gamma \Omega(l) \sigma^2 \right] < 0, \quad \frac{\partial^2 \psi}{\partial l^2} \bigg|_{RR'} = -\frac{\alpha \Omega(l)}{(1 - l)^2} \left[ \frac{1 - \alpha}{1 - \gamma} - \gamma \alpha \sigma^2 \right].
\]

(A.35) is thus decreasing in \( l \) and a sufficient condition for concavity is \( \alpha < 1/2 \). The necessary and sufficient condition for the existence of a unique equilibrium, \( \psi_{pp}(l = 0) > \psi_{RR'}(l = 0) \), is now

\[ -\gamma + \frac{\beta}{A} > -\gamma \frac{1 - \gamma}{2} A \sigma^2. \]  

(A.32'')

Note also that (A.36) schedule is steeper than (A.35) if and only if

\[
\frac{1 + \eta}{\eta} \frac{1}{1 - \gamma} > -\gamma A \sigma^2.
\]  

(A.34')
References


